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Using Sugeno-Weber Triangular Norm-Based Interval Value Spherical Fuzzy Information for Recycled Water

Mehwish Sarfarz[1], Darko Božanić[2*]

[1] Department of Mathematics, Riphah International University (Lahore Campus), 54000 Lahore, Pakistan
[2] Military Academy, University of Defence in Belgrade,11042 Belgrade, Serbia

[*] Correspondence: Darko Božanić (dbozanic@yahoo.com)

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Abstract: This work aims to apply the spherical fuzzy set (SFS), a flexible framework for handling ambiguous human opinions, to improve decision-making processes in recycled water. It specifically looks at the application of Sugeno-Weber (SW) triangular norms in the spherical fuzzy (SF) information domain, providing reliable approximations that are necessary for decision-making. A new class of aggregation operators is presented in this paper. These operators are specifically made for spherical fuzzy information systems and include the interval value spherical fuzzy Sugeno–Weber power weighted average (IVSFSWPA), interval value spherical fuzzy Sugeno–Weber power geometric (IVSFSWPWG), and interval value spherical fuzzy Sugeno–Weber power weighted average (IVSFSWPWA). The realistic features and special cases of these operators are demonstrated, highlighting how well they fit into practical scenarios. A new method for multi-attribute decision-making (MADM) is used for a range of real-world applications with different requirements or characteristics. The efficacy of the recommended methodologies is demonstrated with an example of a recycled water selection process. Additionally, a thorough comparison method is provided to show how the suggested aggregation strategies work and are relevant by contrasting their results with those of the current methods. The study's conclusion highlights the potential contribution of the recommended research to the advancement of decision-making techniques in dynamic and complex environments. It also summarizes its findings and discusses its prospects moving forward.

Keywords: SFS; IVSFSWPA; IVSFSWPG; IVSFSWPWA; IVSFSWPWG; MADM

1 Introduction

The primary goal of MADM problems is to assess or determine the best form that maximizes the limited options while accounting for the preference values that experts or decision-makers provide according to the values of their attributes. Because decision-making in real-world situations is so complex and ambiguous, experts find it uncomfortable and difficult to assign a numerical value to every desire.

Zadeh [1] introduced the fuzzy set (FS) theory. The Zadeh FS theory uses membership grade (MG) to characterize an element's membership in a set. It is occasionally shown that the FS concept's applicability is constrained. For example, all data related to two or more aspects are unrelated to each other. An adjusted version of the FS that can hold complex and indefinite data is called the intuitionistic fuzzy set (IFS), which Atanassov [2] made accessible with the aid of a membership grade (MG) and a non-membership grade (NMG). IFS theory has proven to be useful in solving many problems and has received a lot of support. However, the negative assumption that the sum of MG and NMG is in cell [0,1] limits the size of the IFS dataset. Yager [3] proposed to use of Pythagorean Fuzzy Sets (PyFS), a modification of IFS, to deal with irregular and chaotic data. Similar to IFS, PyFS requires the sum of squares of MG and NMG to be in the range [0,1] and has a limit. If the sum of squares of MG and NMG for a given PyFV is greater than the unit range, for example, if MG is set to 0.5 and NMG is set to 0.9, then MG and NMG cannot track PyFV. Yager [4] introduced q-step orthogonal fuzzy loss (q-ROFS) to solve this problem. It can handle complexities and uncertainties such as those previously mentioned. The MG and NMG pair whose sum of q-powers should be per unit time is called q-ROF value (q-ROFV). Using parameter q, we can select MG and NMG from [0,1] for each pair (MG, NMG). Accordingly $0 \leq MG^q + NMG^q \leq 1$ there is also some denial and avoidance from a human perspective. This means that previous versions of FS models cannot solve complex problems. To solve such problems, Cuong [5] introduced a Picture Fuzzy Set (PFS) containing MG, NMG, and deprivation level (AD) information. The [0,1] range also includes the PFS range. Later, as a generalized version of the picture fuzzy set (PFS), Mahmood et al. [6] assumed global fuzzy set (SFS). Ullah et al. [7] introduced a new concept called global fuzzy set (SFS), which uses q after some modifications.

The idea of using MADM to measure battery life cycle was proposed by Helmers and Weiss [8]. Helmers et al. [9] proposed the concept of measuring the lifetime of an electric vehicle based on MADM data. Lundström and Hellström [10] described how the application can be extended to the evaluation of electric vehicles. A comparison of the main components of vehicle exhaust using natural gas and LPG was made by Stopka et al. [11]. Bartłomiejczyk et al. [12] provided a comparison of charging electric vehicles using MADM-based traction power supply systems. Vitta [13] provided an evaluation of gas-powered vehicles with the environmental characteristics of MADM. Wieckowski et al. [14] proposed the concept of electric vehicle selection in MADM, which is the result of a detailed and complex analysis. Onate et al. [15] proposed an improved version of the MADM optimization life cycle assessment tool for passenger cars. Lim et al. [16] compared the emission content of vehicles using natural gas and liquefied gas. Hao et al. [17] first considered the possibility of using situational analysis to evaluate current and future travelers. A comparison was made by Sarfraz et al. [18] used the concept of pre-assembled workers as IFS. Ulla et al. [19] introduced the concept of Aczel-Alsina values using the IFS function. Sarfraz [20] described the theory of the Dombi hamy mean operator on antiviral masks. Radovanović et al. [21] introduced the application of the hybrid multi-criteria decision-making for the needs of the army.

Radcliffe [22] developed the theory of recycled water of the current status of Australia. Radcliffe and Page [23] introduced the concept of water and recycling situation and future perspective. Posetti et al. [24] gave the theory of recycling water and delivery of the benefit of recycling. Chalmers et al. [25] developed the theory of the source of the regional recycled water program. Tortajada [26] gave the concept of clear water and after we use it. Sarfraz [27] gave the new concept of recycled water with the use of the Schweizer-Sklar prioritized aggregation operators.

Ghodousian et al. [28] gave Sugeno-Weber's TN concept and used the optimization method. Kauers et al. [29] developed the Sugeno-Weber TN theory of FS. Sarkar et al. [30] used the properties of Sugeno-Weber t-norm and t-conorm to provide some good mathematical methods for dealing with uncertain data from the human perspective. Farahbod [31] describe the comparison of T-standard deviations in classification problems. Troiano et al. [32] compared parametric statistical analysis with Sugeno-Weber. Ghodousian [33] gave Sugeno-Weber's theory of two inverse relationships. Saminger-Platz et al. [34] proposed the Sugeno-Weber concept with variable parameters for TN control. Pamucar et al. [35] proposed the Sugeno-Weber theory using the prioritization models. Ghodousian et al. [36] described the Sugeno-Weber scheme for nonlinear problems involving differential equations. Hadzic et al. [37] showed the application of fixed point theory in the field of probability measurement. Ashraf et al. [38] described the theory of Sugeno Weber using the multi-criteria effect on the greenhouse. Ashraf et al. [39] gave the concept of Sugeno Weber aggregation operators using the adaption programming for the media platform.

The following are this article's major contributors:

• Three separate theories provided the foundation for our strong mathematical strategies, which include the operators IVSFSWPA, IVSFSWPG, IVSFSWPWA, and IVSFSWPWG, which have notable characteristics and special cases.

• We evaluated the applicability and effectiveness of the developed AOs using derived strategies, and we devised a sophisticated decision-making process to handle awkward and redundant human opinion data.

• For derived work, we deduce qualities of idempotency, monotonicity, and boundedness.

• A numerical example was also evaluated using the developed methodologies to determine a suitable renewable recycled water under specific prominent features and characteristics.

We looked over some of the most recent information on IVSFS and its SW operating laws in Section 1. The fundamental idea of the Sugeno-Weber T-Norm (SWTN) on IVSFS and the corresponding fundamental operational laws are covered in Section 2. Section 3 contains some laws about SWTN operations on IVSFS. A novel AO with some practical special cases that is based on SWTN on IVSFS operator. We presented the new AOs of the IVSFS operator on MADM in Section 4, and the techniques are described with a numerical example. Besides, to ensure the validity and consistency of the suggested research project, a comparative analysis is set up to contrast the results of the current methodologies with creative strategies. Finally, some concluding thoughts about our research project are provided in Section 5.

2 Methodology

We examine some fundamental concepts of Sugeno-Weber triangular norms, or SFs, and the fundamental rules governing them to develop and validate the proposed research project and validate it.

Definition1. A spherical fuzzy set $\tilde{\alpha}$ of the universe of discourse ω is given by:

$$\alpha = \{\langle \tau, (\mu(\tau), \nu(\tau), \pi(\tau)) \mid \tau \in \omega \rangle\}$$

where $\mu(\tau):\omega\rightarrow[0,1]$, $\nu(\tau):\omega\rightarrow[0,1]$, and $\pi(\tau):\omega\rightarrow[0,1]$ define the degrees of membership, non-membership, and hesitancy, respectively, and

$$0\leq\mu^2(\tau)+\pi^2(\tau)+\nu^2(\tau)\leq 1\quad\forall\tau\in\omega$$

We call $\beta=(\mu,\nu,\pi)$ an SFN (Spherical Fuzzy Number).

Definition2. For any PFV $\alpha=(\mu_\alpha(\tau),v_\alpha(\tau),\pi_\alpha(\tau))$ and $\beta=(\mu_\beta(\tau),v_\beta(\tau),\pi_\beta(\tau))$, with $\lambda_1,\lambda_2,\lambda_3>0$, some necessary operations of the Sugeno-Weber tools are expressed as:

$$\alpha\cup\beta=\left\{\left(\max\left\{\mu_\alpha(\tau),\mu_\beta(\tau)\right\},\min\left\{v_\alpha(\tau),v_\beta(\tau)\right\},\min\left\{1-\left(\max\left\{\mu_\alpha(\tau),\mu_\beta(\tau)\right\}\right)^2+\left(\min\left\{v_\alpha(\tau),v_\beta(\tau)\right\}\right)^2\right\},\max\left\{\pi_\alpha(\tau),\pi_\beta(\tau)\right\}\right)\right\}$$

$$\alpha\cap\beta=\left\{\left(\min\left\{\mu_\alpha(\tau),\mu_\beta(\tau)\right\},\max\left\{v_\alpha(\tau),v_\beta(\tau)\right\},\min\left\{1-\left(\min\left\{\mu_\alpha(\tau),\mu_\beta(\tau)\right\}\right)^2+\left(\max\left\{v_\alpha(\tau),v_\beta(\tau)\right\}\right)^2\right\},\min\left\{\pi_\alpha(\tau),\pi_\beta(\tau)\right\}\right)\right\}$$

$$\alpha\oplus\beta=\left\{\left(\mu_\alpha^2(\tau)+\mu_\beta^2(\tau)-\mu_\alpha^2(\tau)\mu_\beta^2(\tau),v_\alpha(\tau),v_\beta(\tau)(1-\mu_\beta^2(\tau))\pi_\alpha^2(\tau)+(1-\mu_\alpha^2(\tau))\pi_\beta^2(\tau)-\pi_\alpha^2(\tau)\pi_\beta^2(\tau)\right)\right\}$$

$$\alpha\otimes\beta=\left\{\left(\mu_\alpha(\tau),\mu_\beta(\tau),v_\alpha^2(\tau)+v_\beta^2(\tau)-v_\alpha^2(\tau)v_\beta^2(\tau)\right),\left((1-v_\beta^2(\tau))\pi_\alpha^2(\tau)+(1-\mu_\alpha^2(\tau))\pi_\beta^2(\tau)-\pi_\alpha^2(\tau)\pi_\beta^2(\tau)\right)\right\}$$

$$\lambda\cdot\alpha=\left\{\left(1-\left(1-\mu_\alpha^2(\tau)\right)^\lambda\right)^{1/2},\left(\left(1-v_\alpha^2(\tau)\right)^\lambda-\left(1-v_\alpha^2(\tau)\pi_\alpha^2(\tau)\right)^\lambda\right)^{1/2}\right\}$$

λ power of α; $\lambda>0$

$$\alpha^\lambda=\left\{\mu_\alpha^\lambda(\tau),\left(1-\left(1-v_\alpha^2(\tau)\right)^\lambda\right)^{1/2},\left(\left(1-v_\alpha^2(\tau)\right)^\lambda-\left(1-v_\alpha^2(\tau)-\pi_\alpha^2(\tau)\right)^\lambda\right)^{1/2}\right\}$$

Definition3. For the spherical fuzzy sets $\alpha=(\mu_\alpha(\tau),v_\alpha(\tau),\pi_\alpha(\tau))$ and $\beta=(\mu_\beta(\tau),v_\beta(\tau),\pi_\beta(\tau))$, the following holds under the condition $\lambda_1,\lambda_2,\lambda_3>0$:

1. $\alpha\oplus\beta=\beta\oplus\alpha$
2. $\alpha\otimes\beta=\beta\otimes\alpha$
3. $\lambda(\alpha\oplus\beta)=\lambda\alpha\oplus\lambda\beta$
4. $\lambda_1\alpha\oplus\lambda_2\alpha=(\lambda_1\oplus\lambda_2)\alpha$
5. $(\alpha\otimes\beta)^\lambda=\alpha^\lambda\otimes\beta^\lambda$
6. $\alpha^{\lambda_1}\otimes\alpha^{\lambda_2}=\alpha^{\lambda_1+\lambda_2}$

Definition4. An interval-valued spherical fuzzy set α of the universe of discourse ω is defined as follows:

$$\alpha=\left\{\tau,\left([\mu^l(\tau),\mu^u(\tau)],[\nu^l(\tau),v^u(\tau)],[\pi^l(\tau),\pi^u(\tau)]\right)\mid\tau\in\omega\right\}$$

where

$$0\leq\left((\mu^u(\tau))^2+(v^u(\tau))^2+(\pi^u(\tau))^2\right)\leq 1,\quad 0\leq v^l(\tau)\leq v^u(\tau)$$

where, $\mu(\tau)=[\mu^l(\tau),\mu^u(\tau)]$, $v(\tau)=[v^l(\tau),v^u(\tau)]$, $\pi(\tau)=[\pi^l(\tau),\pi^u(\tau)]$.

The expression $r(\tau)=[r^l(\tau),r^u(\tau)]=\left[\sqrt{1-((\mu^u(\tau))^2+(v^u(\tau))^2+(\pi^u(\tau))^2)},\sqrt{1-\left((\mu^l(\tau))^2+(v^l(\tau))^2+(\pi^l(\tau))^2\right)}\right]$ is termed as DR. We call $\alpha=(\mu,v,\pi)=\mu^l(\tau),\mu^u(\tau)$ an IVSF number.

2.1 Operations on Interval-Valued Spherical Fuzzy Sets

Definition5. Let $\alpha=\langle[\mu^l(\tau),\mu^u(\tau)],[v^l(\tau),v^u(\tau)],[\pi^l(\tau),\pi^u(\tau)]\rangle$, $\alpha_1=\langle[\mu_1^l(\tau),\mu_1^u(\tau)],[v_1^l(\tau),v_1^u(\tau)],[\pi_1^l(\tau),\pi_1^u(\tau)]\rangle$, and $\alpha_2=\langle[\mu_2^l(\tau),\mu_2^u(\tau)],[v_2^l(\tau),v_2^u(\tau)],[\pi_2^l(\tau),\pi_2^u(\tau)]\rangle$ be three interval-valued spherical fuzzy sets (IVSFS). Then:

$$\alpha_1\cup\alpha_2=\langle\left[\max\{\mu_1^l(\tau),\mu_2^l(\tau)\},\max\{\mu_1^u(\tau),\mu_2^u(\tau)\}\right],\left[\min\{v_1^l(\tau),v_2^l(\tau)\},\min\{v_1^u(\tau),v_2^u(\tau)\}\right],\left[\min\{\pi_1^l(\tau),\pi_2^l(\tau)\},\min\{\pi_1^u(\tau),\pi_2^u(\tau)\}\right]\rangle$$

$$\alpha_1\cap\alpha_2=\langle\left[\min\{\mu_1^l(\tau),\mu_2^l(\tau)\},\min\{\mu_1^u(\tau),\mu_2^u(\tau)\}\right],\left[\max\{v_1^l(\tau),v_2^l(\tau)\},\max\{v_1^u(\tau),v_2^u(\tau)\}\right],\left[\min\{\pi_1^l(\tau),\pi_2^l(\tau)\},\min\{\pi_1^u(\tau),\pi_2^u(\tau)\}\right]\rangle$$

$$\alpha_1\oplus\alpha_2=\langle\left[\sqrt{(\mu_1^l(\tau))^2+(\mu_2^l(\tau))^2-(\mu_1^l(\tau))^2(\mu_2^l(\tau))^2},\sqrt{(\mu_1^u(\tau))^2+(\mu_2^u(\tau))^2-(\mu_1^u(\tau))^2(\mu_2^u(\tau))^2}\right],\left[v_1^l(\tau)v_2^l(\tau),v_1^u(\tau)v_2^u(\tau)\right],\left[\pi_1^l(\tau)\pi_2^l(\tau),\pi_1^u(\tau)\pi_2^u(\tau)\right]\rangle$$

$$\alpha_1 \otimes \alpha_2 = \langle \left[\mu_1^l(\tau)\mu_2^l(\tau), \mu_1^u(\tau)\mu_2^u(\tau)\right], \left[\sqrt{(v_1^l(\tau))^2 + (v_2^l(\tau))^2 - (v_1^l(\tau))^2(v_2^l(\tau))^2}, \sqrt{(v_1^u(\tau))^2 + (v_2^u(\tau))^2 - (v_1^u(\tau))^2(v_2^u(\tau))^2}\right], \left[\sqrt{(\pi_1^l(\tau))^2 + (\pi_2^l(\tau))^2 - (\pi_1^l(\tau))^2(\pi_2^l(\tau))^2}, \sqrt{(\pi_1^u(\tau))^2 + (\pi_2^u(\tau))^2 - (\pi_1^u(\tau))^2(\pi_2^u(\tau))^2}\right] \rangle$$

$$\lambda \cdot \alpha = \left\langle \left[\sqrt{1 - \left(1 - (\mu^l(\tau))^2\right)^\lambda}, \ \sqrt{1 - \left(1 - (\mu^u(\tau))^2\right)^\lambda}\right], \left[(v^l(\tau))^\lambda, (v^u(\tau))^\lambda\right], \left[(\pi^l(\tau))^\lambda, (\pi^u(\tau))^\lambda\right] \right\rangle$$

$$\alpha^\lambda = \left\langle \left[(\mu^l(\tau))^\lambda, (\mu^u(\tau))^\lambda\right], \quad \left[\sqrt{1 - (1 - (v^l(\tau))^2)^\lambda}, \quad \sqrt{1 - (1 - (v^u(\tau))^2)^\lambda}\right], \left[\sqrt{1 - (1 - (\pi^l(\tau))^2)^\lambda}, \sqrt{1 - (1 - (\pi^u(\tau))^2)^\lambda}\right] \right\rangle$$

Definition6. Let $\alpha = \langle [\mu_j^l(\tau), \mu_j^u(\tau)], [v_j^l(\tau), v_j^u(\tau)], [\pi_j^l(\tau), \pi_j^u(\tau)] \rangle$ be a collection of Interval-Valued Spherical Fuzzy Sets (IVSFS) with respect to $\omega_j = (\omega_1, \omega_2, \ldots, \omega_n)$; $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Then the Interval-Valued Spherical Weighted Arithmetic Mean (IVSWAM) is defined as:

$$\text{IVSWAM}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \omega_1 \cdot \alpha_1 \oplus \omega_2 \cdot \alpha_2 \oplus \cdots \oplus \omega_n \cdot \alpha_n$$

$$\text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \left\langle \begin{array}{l} \left[\sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_j^l(\tau))^2\right)^{\omega_j}}, \sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_j^u(\tau))^2\right)^{\omega_j}}\right], \\ \left[\prod_{j=1}^n \left(v_j^l(\tau)\right)^{\omega_j}, \prod_{j=1}^n \left(v_j^u(\tau)\right)^{\omega_j}\right], \\ \left[\sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_j^l(\tau))^2 - (\pi_j^l(\tau))^2\right)^{\omega_j}}, \right. \\ \left. \sqrt{1 - \prod_{j=1}^n \left(1 - (\mu_j^u(\tau))^2 - (\pi_j^u(\tau))^2\right)^{\omega_j}}\right] \end{array} \right\rangle$$

Definition7. Let $\alpha = \langle [\mu_j^l(\tau), \mu_j^u(\tau)], [v_j^l(\tau), v_j^u(\tau)], [\pi_j^l(\tau), \pi_j^u(\tau)] \rangle$ be a collection of Interval-Valued Spherical Fuzzy Sets (IVSFS) concerning $\omega_j = (\omega_1, \omega_2, \ldots, \omega_n)$; $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Then the weighted Interval-Valued Spherical Weighted Arithmetic Mean (IVSWAM_w) is defined as:

$$\text{IVSWAM}_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \omega_1 \cdot \alpha_1 \oplus \omega_2 \cdot \alpha_2 \oplus \cdots \oplus \omega_n \cdot \alpha_n$$

$$\text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \left\langle \begin{array}{l} \left[\prod_{j=1}^n (\mu_j^l(\tau))^{\omega_j}, \prod_{j=1}^n (\mu_j^u(\tau))^{\omega_j}\right], \\ \left[\sqrt{1 - \prod_{j=1}^n \left(1 - (v_j^l(\tau))^2\right)^{\omega_j}}, \sqrt{1 - \prod_{j=1}^n \left(1 - (v_j^u(\tau))^2\right)^{\omega_j}}\right], \\ \left[\sqrt{\prod_{j=1}^n \left(1 - (v_j^l(\tau))^2\right)^{\omega_j} - \prod_{j=1}^n \left(1 - (v_j^l(\tau))^2 - (\pi_j^l(\tau))^2\right)^{\omega_j}}, \right. \\ \left. \sqrt{\prod_{j=1}^n \left(1 - (v_j^u(\tau))^2\right)^{\omega_j} - \prod_{j=1}^n \left(1 - (v_j^u(\tau))^2 - (\pi_j^u(\tau))^2\right)^{\omega_j}}\right] \end{array} \right\rangle$$

3 Operation on Sugeno-Weber Triangular Norm Based on IVSF Information

Definition8. For three IVSFVs, $\mathbf{S} = ([\mu^l(\tau), \mu^u(\tau)], [\pi^l(\tau), \pi^u(\tau)], [v^l(\tau), v^u(\tau)])$, $\theta_1 = ([\mu_1^l(\tau), \mu_1^u(\tau)], [\pi_1^l(\tau), \pi_1^u(\tau)], [v_1^l(\tau), v_1^u(\tau)])$, and $\theta_2 = ([\mu_2^l(\tau), \mu_2^u(\tau)], [\pi_2^l(\tau), \pi_2^u(\tau)], [v_2^l(\tau), v_2^u(\tau)])$, the basic operations of Sugeno-Weber triangular norms are given by:

$$\theta_1 \oplus \theta_2 = \left(\begin{array}{l} \left[\sqrt{\frac{\mu_1^l(\tau)^2 + \mu_2^l(\tau)^2 - \frac{\vartheta}{1+\vartheta}\mu_1^l(\tau)^2\mu_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\mu_1^u(\tau)^2 + \mu_2^u(\tau)^2 - \frac{\vartheta}{1+\vartheta}\mu_1^u(\tau)^2\mu_2^u(\tau)^2}{1+\vartheta}}\right] \\ \left[\sqrt{\frac{\pi_1^l(\tau)^2 + \pi_2^l(\tau)^2 - 1 + \vartheta\pi_1^l(\tau)^2\pi_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\pi_1^u(\tau)^2 + \pi_2^u(\tau)^2 - 1 + \vartheta\pi_1^u(\tau)^2\pi_2^u(\tau)^2}{1+\vartheta}}\right] \\ \left[\sqrt{\frac{\nu_1^l(\tau)^2 + \nu_2^l(\tau)^2 - 1 + \vartheta\nu_1^l(\tau)^2\nu_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\nu_1^u(\tau)^2 + \nu_2^u(\tau)^2 - 1 + \vartheta\nu_1^u(\tau)^2\nu_2^u(\tau)^2}{1+\vartheta}}\right] \end{array} \right)$$

$$\theta_1 \otimes \theta_2 = \left(\begin{array}{l} \left[\sqrt{\frac{\mu_1^l(\tau)^2 + \mu_2^l(\tau)^2 - 1 + \vartheta\mu_1^l(\tau)^2\mu_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\mu_1^u(\tau)^2 + \mu_2^u(\tau)^2 - 1 + \vartheta\mu_1^u(\tau)^2\mu_2^u(\tau)^2}{1+\vartheta}}\right] \\ \left[\sqrt{\frac{\pi_1^l(\tau)^2 + \pi_2^l(\tau)^2 - \frac{\vartheta}{1+\vartheta}\pi_1^l(\tau)^2\pi_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\pi_1^u(\tau)^2 + \pi_2^u(\tau)^2 - \frac{\vartheta}{1+\vartheta}\pi_1^u(\tau)^2\pi_2^u(\tau)^2}{1+\vartheta}}\right] \\ \left[\sqrt{\frac{\nu_1^l(\tau)^2 + \nu_2^l(\tau)^2 - \frac{\vartheta}{1+\vartheta}\nu_1^l(\tau)^2\nu_2^l(\tau)^2}{1+\vartheta}}, \sqrt{\frac{\nu_1^u(\tau)^2 + \nu_2^u(\tau)^2 - \frac{\vartheta}{1+\vartheta}\nu_1^u(\tau)^2\nu_2^u(\tau)^2}{1+\vartheta}}\right] \end{array} \right)$$

$$\Delta\theta = \left(\begin{array}{c} \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^l(\tau))^2\,\frac{\vartheta}{1+\vartheta}\right)^{\Delta}\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^u(\tau))^2\,\frac{\vartheta}{1+\vartheta}\right)^{\Delta}\right)}\right] \\ \left[\sqrt{\frac{(1+\vartheta)}{\vartheta}\left(1-\left(\frac{1+\vartheta(\pi^l(\tau))^2}{1+\vartheta}\right)^{\Delta}-1\right)}, \sqrt{\frac{(1+\vartheta)}{\vartheta}\left(1-\left(\frac{1+\vartheta(\pi^u(\tau))^2}{1+\vartheta}\right)^{\Delta}-1\right)}\right] \\ \left[\sqrt{\frac{(1+\vartheta)}{\vartheta}\left(1-\left(\frac{1+\vartheta(\nu^l(\tau))^2}{1+\vartheta}\right)^{\Delta}-1\right)}, \sqrt{\frac{(1+\vartheta)}{\vartheta}\left(1-\left(\frac{1+\vartheta(\nu^u(\tau))^2}{1+\vartheta}\right)^{\Delta}-1\right)}\right] \end{array} \right)$$

Definition9. Consider a class of IVSFVs $\Delta_i = ([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]), i = 1, 2, \ldots, \eta$. The IVSFSWPA operator is characterized as follows:

$$\text{IVSFSWPA}(\Delta_1, \Delta_2, \ldots, \Delta_\eta) = \bigoplus_{i=1}^{\eta} \nabla_i \Delta_i$$

where

$$\nabla_i = \frac{1 + \mathcal{A}(\Delta_i)}{\sum_{i=1}^{\eta}(1 + \mathcal{A}(\Delta_i))}$$

and

$$\mathcal{A}(\Delta_i) = \sum_{i=1}^{\eta} \text{supp}(\Delta_i, \Delta_\tau)$$

Theorem1. Consider a class of IVSFVs $\Delta_i = ([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]), i = 1, 2, \ldots, \eta$. The accumulated value of the IVSFSWPA operator is still an IVSFV, so we have the following expression:

$$\text{IVSFSWPA}(\Delta_1, \Delta_2, \ldots, \Delta_\eta) = \left(\begin{array}{c} \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\eta}\left(1-(\mu_i^l(\tau))^2\,\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\eta}\left(1-(\mu_i^u(\tau))^2\,\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\right] \\ \left[\sqrt{\frac{1}{\vartheta}\left(1+\vartheta\prod_{i=1}^{\eta}\left(1-\frac{\pi_i^l(\tau)^2}{1+\vartheta}\right)^{\nabla_i}\right)}, \sqrt{\frac{1}{\vartheta}\left(1+\vartheta\prod_{i=1}^{\eta}\left(1-\frac{\pi_i^u(\tau)^2}{1+\vartheta}\right)^{\nabla_i}\right)}\right] \\ \left[\sqrt{\frac{1}{\vartheta}\left(1+\vartheta\prod_{i=1}^{\eta}\left(1-\frac{\nu_i^l(\tau)^2}{1+\vartheta}\right)^{\nabla_i}\right)}, \sqrt{\frac{1}{\vartheta}\left(1+\vartheta\prod_{i=1}^{\eta}\left(1-\frac{\nu_i^u(\tau)^2}{1+\vartheta}\right)^{\nabla_i}\right)}\right] \end{array} \right)$$

Proof. We want to prove the following expression using induction for $n = 2$. We can write:

$$\nabla_1\mathbf{\Delta_1} = \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu_{(1)}^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_1}\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu_{(1)}^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_1}\right)}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(\pi_{(1)}^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_1}-1\right)\frac{1}{\vartheta}}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(\pi_{(1)}^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_1}-1\right)\frac{1}{\vartheta}}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(v_{(1)}^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_1}-1\right)\frac{1}{\vartheta}}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(v_{(1)}^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_1}-1\right)\frac{1}{\vartheta}}\right]$$

$$\nabla_2\mathbf{\Delta_2} = \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu_{(2)}^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_2}\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu_{(2)}^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_2}\right)}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(\pi_{(2)}^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_2}-1\right)\frac{1}{\vartheta}}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(\pi_{(2)}^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_2}-1\right)\frac{1}{\vartheta}}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(v_{(2)}^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_2}-1\right)\frac{1}{\vartheta}}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(\left(\frac{\vartheta(v_{(2)}^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_2}-1\right)\frac{1}{\vartheta}}\right]$$

The Interval-Valued Hesitant Fuzzy Sum with Weights and Parameters is given by:

$$\text{IVSFSWPÆ}(\Delta_1, \Delta_2) = \bigoplus_{i=1}^{2} \nabla_i \Delta_i$$

Thus:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2)=\begin{bmatrix} \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{2}\left(1-(\mu_i^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} & \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{2}\left(1-(\mu_i^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{2}\left(\frac{\vartheta(\pi_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} & \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{2}\left(\frac{\vartheta(\pi_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{2}\left(\frac{\vartheta(v_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} & \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{2}\left(\frac{\vartheta(v_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \end{bmatrix}$$

Assume the above expression is true for $n=k$:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_k)=\bigoplus_{i=1}^{k}\nabla_i\Delta_i$$

We need to prove it for $n=k+1$:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_{k+1})=\text{IVSFSWPA}(\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_k),\Delta_{k+1})$$

Thus:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_{k+1})=\begin{bmatrix} \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{k+1}\left(1-(\mu_i^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{k+1}\left(1-(\mu_i^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{k+1}\left(\frac{\vartheta(\pi_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{k+1}\left(\frac{\vartheta(\pi_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{k+1}\left(\frac{\vartheta(v_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(\prod_{i=1}^{k+1}\left(\frac{\vartheta(v_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)\frac{1}{\vartheta}} \end{bmatrix}$$

Hence proved.

Theorem2. Consider a class of Interval-Valued Soft Fuzzy Variables (IVSFVs):

$$\Delta_i=\left([\mu_i^l(\tau),\mu_i^u(\tau)],[\pi_i^l(\tau),\pi_i^u(\tau)],[v_i^l(\tau),v_i^u(\tau)]\right)$$

where $i=1,2,\ldots,\nu$ are identical, i.e., $\Delta_i=\Delta$. Then we have:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\Delta$$

Proof. Since all IVSFVs $\Delta_i=\left([\mu_i^l(\tau),\mu_i^u(\tau)],[\pi_i^l(\tau),\pi_i^u(\tau)],[v_i^l(\tau),v_i^u(\tau)]\right)$, where $i=1,2,\ldots,\nu$, are identical, i.e., $\Delta_i=\Delta$, we have:

$$\text{IVSFSWPA}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\mu_i^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)},\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\mu_i^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\right]$$

$$=\left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\pi_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)},\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\pi_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)}\right]$$

$$=\left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(v_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)},\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(v_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)}\right]$$

$$=\left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\nabla_i}\right)},\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\nabla_i}\right)}\right]$$

$$= \left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(\pi^l(\tau))^2+1}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\omega_i}-1\right)}, \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(\pi^u(\tau))^2+1}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\omega_i}-1\right)}\right]$$

$$= \left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(v^l(\tau))^2+1}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\omega_i}-1\right)}, \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(v^u(\tau))^2+1}{1+\vartheta}\right)^{\sum_{i=1}^{\nu}\omega_i}-1\right)}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\left(1-(\mu^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)\right)}\right]$$

$$= \left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(\pi^l(\tau))^2+1}{1+\vartheta}\right)-1\right)}, \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(\pi^u(\tau))^2+1}{1+\vartheta}\right)-1\right)}\right]$$

$$= \left[\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(v^l(\tau))^2+1}{1+\vartheta}\right)-1\right)}, \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\left(\frac{\vartheta(v^u(\tau))^2+1}{1+\vartheta}\right)-1\right)}\right]$$

$$= \left[\sqrt{\frac{1+\vartheta}{\vartheta}\left((\mu^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)}, \sqrt{\frac{1+\vartheta}{\vartheta}\left((\mu^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)}\right] = \left[\sqrt{\frac{1}{\vartheta}\left(\vartheta(\pi^l(\tau))^2\right)}, \sqrt{\frac{1}{\vartheta}\left(\vartheta(\pi^u(\tau))^2\right)}\right]$$

$$= \left[\sqrt{\frac{1}{\vartheta}\left(\vartheta(v^l(\tau))^2\right)}, \sqrt{\frac{1}{\vartheta}\left(\vartheta(v^u(\tau))^2\right)}\right] = \left[\sqrt{(\mu(\tau))^2\frac{\vartheta}{1+\vartheta}}, \sqrt{(\mu(\tau))^2\frac{\vartheta}{1+\vartheta}}\right]$$

$$= \left[\sqrt{\vartheta(\pi(\tau))^2}, \sqrt{\vartheta(\pi(\tau))^2}\right] = \left[\sqrt{\vartheta(v(\tau))^2}, \sqrt{\vartheta(v(\tau))^2}\right] = (\mu(\tau), \pi(\tau), v(\tau)) = \Delta$$

Hence proved.

Theorem3. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$, implying that $\Delta_i = \Delta$. Then we obtain:

$$\text{IVSFWPOA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) = \Delta$$

Theorem4. Consider any two sets of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$ and $\Delta_i' = \left([\mu_i'^l(\tau), \mu_i'^u(\tau)], [\pi_i'^l(\tau), \pi_i'^u(\tau)], [v_i'^l(\tau), v_i'^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta_i \leq \Delta_i'$, then we get:

$$\text{IVSFWPOA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq q - \text{IVSFWPOA}(\Delta_1', \Delta_2', \ldots, \Delta_n')$$

Definition10. For a class of IVSFVs $\Delta_i = \left([\mu_i'^l(\tau), \mu_i'^u(\tau)], [\pi_i'^l(\tau), \pi_i'^u(\tau)], [v_i'^l(\tau), v_i'^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$, the IVSFSWOPA operator is characterized as follows:

$$\text{IVSFSWOPA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) = \bigoplus_{i=1}^{\nu} \nabla_i \Delta_{\rho(i)}$$

where

$$\nabla_i = \frac{1+\xi(\Delta_i)}{\sum_{i=1}^{\nu}(1+\xi(\Delta_i))} \quad \text{and} \quad \xi(\Delta_i) = \sum_{\substack{i=1\\ i\neq\tau}}^{\nu} \text{supp}(\Delta_i, \Delta_\tau)$$

and let $\rho(1), \rho(2), \ldots, \rho(\nu)$ be a permutation of $i = 1, 2, \ldots, \nu$ such that $\Delta_{\rho(i-1)} \geq \Delta_i$.

Definition11. For a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$, the IVSFSWPG operator is characterized as follows:

$$\text{IVSFSWPG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) = \bigotimes_{i=1}^{\nu} \Delta_i^{\nabla_i}$$

where

$$\nabla_i = \frac{1+\xi(\Delta_i)}{\sum_{i=1}^{\nu}(1+\xi(\Delta_i))} \quad \text{and} \quad \xi(\Delta_i) = \sum_{\substack{i=1\\ i\neq\tau}}^{\nu} \text{supp}(\Delta_i, \Delta_\tau)$$

Theorem5. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. The accumulated value of the IVSFSWOPA operator is still an IVFV, so we get the following:

$$\text{IVSFSWOPA}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\left[\begin{array}{c}\sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\mu^l_{\rho(i)}(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)},\\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\mu^u_{\rho(i)}(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\pi^l_{\rho(i)}(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)},\\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\pi^u_{\rho(i)}(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)}\\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(v^l_{\rho(i)}(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)},\\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(v^u_{\rho(i)}(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)}\end{array}\right]$$

where

$$\nabla_i=\frac{1+\xi(\Delta_i)}{\sum_{i=1}^{\nu}(1+\xi(\Delta_i))}$$

and

$$\xi(\Delta_i)=\sum_{\substack{i=1\\ i\neq\tau}}^{\nu}\text{supp}(\Delta_i,\Delta_\tau)$$

and $\rho(1),\rho(2),\ldots,\rho(\nu)$ is a permutation of $i=1,2,\ldots,\nu$ such that $\Delta_{\rho(i-1)}\geq\Delta_i$.

Theorem6. Consider a class of IVSFVs $\Delta_i=\left([\mu^l_i(\tau),\mu^u_i(\tau)],[\pi^l_i(\tau),\pi^u_i(\tau)],[v^l_i(\tau),v^u_i(\tau)]\right)$, where $i=1,2,\ldots,\nu$. The accumulated value of the IVSFSWPG operator is still an IVSFV, so we get the following:

$$\text{IVSFSWPG}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\left[\begin{array}{c}\sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\mu^l_i(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)}\\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\mu^u_i(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i}-1\right)},\\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\pi^l_i(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\pi^u_i(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)},\\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(v^l_i(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(v^u_i(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)}\end{array}\right]$$

Theorem7. Consider a class of IVSFVs $\Delta_i=\left([\mu^l_i(\tau),\mu^u_i(\tau)],[\pi^l_i(\tau),\pi^u_i(\tau)],[v^l_i(\tau),v^u_i(\tau)]\right)$, where $i=1,2,\ldots,\nu$, which implies that $\Delta_i=\Delta$. Then we obtain:

$$\text{IVSFSWPG}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\Delta$$

Theorem8. Consider two sets of IVSFVs $\Delta_i=\left([\mu^l_i(\tau),\mu^u_i(\tau)],[\pi^l_i(\tau),\pi^u_i(\tau)],[v^l_i(\tau),v^u_i(\tau)]\right)$ and $\Delta'_i=\left([\mu'^l_i(\tau),\mu'^u_i(\tau)],[\pi'^l_i(\tau),\pi'^u_i(\tau)],[v'^l_i(\tau),v'^u_i(\tau)]\right)$, where $i=1,2,\ldots,\nu$. If $\Delta_i\leq\Delta'_i$, then we get:

$$\text{IVSFSWPG}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)\leq q-\text{IVSFSWPG}(\Delta'_1,\Delta'_2,\ldots,\Delta'_\nu).$$

Definition12. For a class of IVSFVs $\Delta_i=\left([\mu^l_i(\tau),\mu^u_i(\tau)],[\pi^l_i(\tau),\pi^u_i(\tau)],[v^l_i(\tau),v^u_i(\tau)]\right)$, where $i=1,2,\ldots,\nu$, the IVSFSWPOG operator is characterized as follows:

$$\text{IVSFSWPOG}(\Delta_1,\Delta_2,\ldots,\Delta_\nu)=\bigoplus_{i=1}^{\nu}\Delta_{\rho(i)}^{\nabla_i}$$

where

$$\nabla_i = \frac{1+\xi(\Delta_i)}{\sum_{i=1}^{\nu}(1+\xi(\Delta_i))}$$

and

$$\xi(\Delta_i) = \sum_{\substack{i=1 \\ i\neq\tau}}^{\nu} \text{supp}(\Delta_i, \Delta_\tau)$$

and $\rho(1), \rho(2), \ldots, \rho(\nu)$ is a permutation of $i = 1, 2, \ldots, \nu$ such that $\Delta_{\rho(i-1)} \geq \Delta_i$.

Theorem9. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. The accumulated value of the IVSFSWPOG operator is still an IVSFV, so we get the following:

$$\text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) = \begin{bmatrix} \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\mu_i^l(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i} - 1\right)} \\ \sqrt{\frac{1}{\vartheta}\left((1+\vartheta)\prod_{i=1}^{\nu}\left(\frac{\vartheta(\mu_i^u(\tau))^2+1}{1+\vartheta}\right)^{\nabla_i} - 1\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\pi_i^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(\pi_i^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(v_i^l(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \\ \sqrt{\frac{1+\vartheta}{\vartheta}\left(1-\prod_{i=1}^{\nu}\left(1-(v_i^u(\tau))^2\frac{\vartheta}{1+\vartheta}\right)^{\nabla_i}\right)} \end{bmatrix}$$

Theorem10. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$, which implies that $\Delta_i = \Delta$. Then we obtain:

$$\text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) = \Delta$$

Theorem11. Consider any two sets of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$ and $\Delta_i' = \left([\mu_i'^l(\tau), \mu_i'^u(\tau)], [\pi_i'^l(\tau), \pi_i'^u(\tau)], [v_i'^l(\tau), v_i'^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta_i \leq \Delta_i'$, then we get:

$$\text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq q - \text{IVSFSWPOG}(\Delta_1', \Delta_2', \ldots, \Delta_\nu')$$

Theorem12. For any two sets of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$ and $\Delta_i' = \left([\mu_i'^l(\tau), \mu_i'^u(\tau)], [\pi_i'^l(\tau), \pi_i'^u(\tau)], [v_i'^l(\tau), v_i'^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta_i \leq \Delta_i'$, then we get:

$$\text{IVSFSWPA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq q - \text{IVSFSWPA}(\Delta_1', \Delta_2', \ldots, \Delta_\nu')$$

Theorem13. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

Theorem14. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPOA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

Theorem15. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

Theorem16. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPOG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

Theorem17. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPWA}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

Theorem18. Consider a class of IVSFVs $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right)$, where $i = 1, 2, \ldots, \nu$. If $\Delta^- = (\min\{\mu_i^l, \mu_i^l\}, \max\{\pi_i^l, \pi_i^u\}, \max\{v_i^l, v_i^u\})$ and $\Delta^+ = (\max\{\mu_i^l, \mu_i^u\}, \min\{\pi_i^l, \pi_i^u\}, \min\{v_i^l, v_i^u\})$, then we get:

$$\Delta^- \leq \text{IVSFSWPWG}(\Delta_1, \Delta_2, \ldots, \Delta_\nu) \leq \Delta^+$$

4 Assessment of MSCD Problems Based on Aggregation Operator

This section evaluates uncertainty in human opinions by applying the structure of the IVFSWPWA and IVFSWPWG Operators. The SF information covers extensive information about any object in the form of membership terms (MTs) non-membership terms (NMTs) and hesitancy terms (ATs). Consider a finite collection of an alternative denoted by Consider a finite collection of alternatives denoted by $\partial = (\partial_1, \partial_2, \ldots, \partial_\nu)$ and a set of finite attributes $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$. Each attribute is assigned a weight $F = (F_1, F_2, \ldots, F_\nu)$ such that $F_i \geq 0$ and $\sum_{i=1}^{\nu} F_i = 1$.

The decision maker takes information in the form of soft sets (SF) where $0 \leq \mu^2(\tau) + \pi^2(\tau) + v^2(\tau) \leq 1, \forall \tau \in \Omega$, and lists the given information as $\Delta_i = \left([\mu_i^l(\tau), \mu_i^u(\tau)], [\pi_i^l(\tau), \pi_i^u(\tau)], [v_i^l(\tau), v_i^u(\tau)]\right), i = 1, 2, \ldots, \nu$.

In a standard decision matrix, we have $\partial = (\mathcal{C}_{ji})$.

The decision maker proposes an innovative algorithm for the multi-attribute decision-making (MADM) technique as follows:

4.1 Algorithm

Step 1: The decision maker acquires information about any realistic object in the form of IF information. And states it as a standard decision matrix.

Step 2: Collected information having two types of attributes such as beneficial and non-beneficial. If given information contains more than one type of attribute, we need to transform the standard decision matrix into a normalized decision matrix; otherwise, this technique is necessary.

Step 3: Compute support by using the following expression:

$$\text{supp}(\Delta_{ij}, \Delta_{jk}) = 1 - D(\Delta_{ij}, \Delta_{jk})$$

$$D(\Delta_i, \Delta_\tau) = \frac{1}{6}\left(\left|(\mu_i^l)^2 - (\mu_k^l)^2\right| + \left|(\mu_j^u)^2 - (\mu_k^u)^2\right| + \left|(v_i^l)^2 - (v_j^l)^2\right| + \left|(v_j^u)^2 - (v_k^u)^2\right| + \left|(\pi_i^l)^2 - (\pi_j^l)^2\right| + \left|(\pi_j^u)^2 - (\pi_k^u)^2\right|\right)$$

Step 4: Calculate the degree of weighted support:

$$\text{Æ}(\Delta) = \sum_{\substack{i=1 \\ i\neq\tau}}^{\nu} E_i \, \text{supp}(\Delta_i, \Delta_\tau)$$

where $E = (E_1, E_2, \ldots, E_\nu)$, $E_i > 0$, and $\sum_{\substack{i=1 \\ i\neq\tau}}^{\nu} E_i = 1$.

Step 5: Investigate the degree of the support:

$$\Phi = \frac{E_i(1 + \text{Æ}(\Delta_i))}{\sum_{\substack{i=1 \\ i\neq\tau}}^{\nu} E_i(1 + \text{Æ}(\Delta_i))}$$

Step 6: The provided data is consolidated through the utilization of the IVSFSWPA, IVSFSWPG, IVSFSWPWA, and IVSFSWPWG operators.

Step 7: To identify practical optimal choices, the score values of each individual are thoroughly examined.

Step 8: The ranking and ordering technique is applied to reorganize all score values.

4.2 Experimental Case Study

Recycling water is even more important in areas where water is scarce. It is obtained by purifying wastewater and reusing it for non-useful purposes. One popular option is a larger option that can be used to collect and clean household wastewater from showers, sinks, and laundry rooms for use in toilets and water. Roofs that collect and store runoff water and are used for landscaping or groundwater irrigation purposes. Rainwater harvesting and reuse can replenish urban water supplies for non-renewable water resources, groundwater recharge, or runoff, thus reducing freshwater supplies and reducing stormy winds. Treatment and recycling facilities and commercial and residential wastewater are changing with the use of new treatment technologies. Figure 1 shows the process of recycling of water by T-spherical.

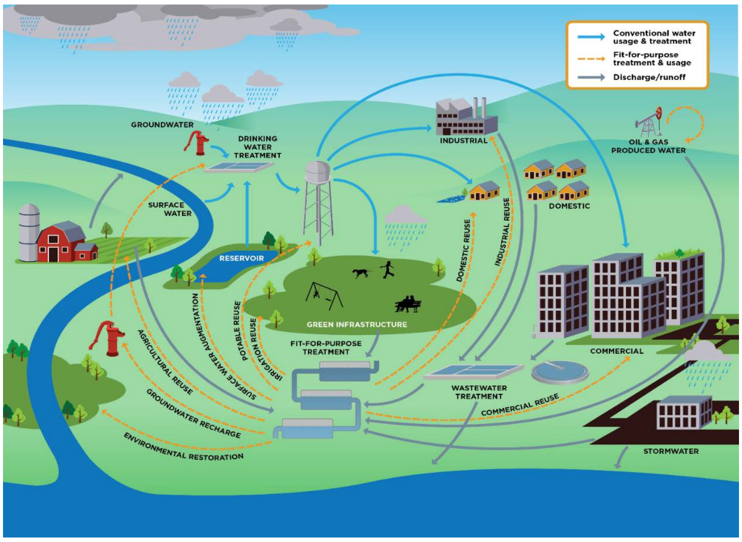

Figure 1. Recycling water

Water recovered from these facilities can be used in industrial processes, landscape irrigation and environmental improvement. This reduces the amount of wastewater discharged into natural waters and reduces dependence on fresh water. Additionally, desalination of brackish water using reverse osmosis or other desalination technologies is a viable solution in areas where infiltration of saltwater into soil aquifers is a problem. This ensures the safety of fresh water. These options vary in cost-effectiveness, energy consumption, environmental impact, water quality and energy efficiency; this highlights the need for careful analysis to determine the best option for specific situations and applications. We currently have five different options:

A_1: Storm water capture and reuse
A_2: Rainwater harvesting
A_3: Grey water recycling system
A_4: Treatment and reclamation of wastewater
A_5: Making brackish water desalinate

The decision making assessment of these recycled water is based on four different types of attributes such as:

G_1: Economy of scale
G_2: Effects on the environment
G_3: Capability to grow
G_4: Water quality

The decision-maker considers the aforementioned attributes as they evaluate data pertaining to recycled water. Theoretically, the weight of the criteria (0.10, 0.25, 0.35, 0.30) is assigned by the decision-maker. By using the recommended approaches, the decision-maker evaluates a suitable building material in accordance with the recommended MADM process algorithm values in Tables 1 and 2.

Table 1. Details about the recycled water suppliers

Supplier	G_1	G_2	G_3	G_4
A_1	([0.27, 0.35], [0.46, 0.55], [0.54, 0.56])	([0.43, 0.51], [0.25, 0.33], [0.23, 0.33])	([0.14, 0.28], [0.43, 0.55], [0.60, 0.66])	([0.45, 0.47], [0.32, 0.46], [0.14, 0.17])
A_2	([0.33, 0.88], [0.21, 0.34], [0.10, 0.15])	([0.11, 0.15], [0.16, 0.27], [0.44, 0.77])	([0.25, 0.33], [0.47, 0.49], [0.14, 0.17])	([0.22, 0.32], [0.20, 0.27], [0.13, 0.19])
A_3	([0.54, 0.61], [0.32, 0.35], [0.24, 0.35])	([0.36, 0.42], [0.14, 0.18], [0.22, 0.24])	([0.48, 0.59], [0.15, 0.18], [0.21, 0.23])	([0.44, 0.49], [0.46, 0.56], [0.10, 0.15])
A_4	([0.10, 0.15], [0.44, 0.55], [0.43, 0.47])	([0.15, 0.19], [0.43, 0.52], [0.11, 0.17])	([0.12, 0.18], [0.78, 0.82], [0.18, 0.21])	([0.14, 0.17], [0.44, 0.56], [0.17, 0.24])
A_5	([0.20, 0.24], [0.36, 0.41], [0.45, 0.57])	([0.52, 0.57], [0.44, 0.48], [0.13, 0.22])	([0.44, 0.48], [0.16, 0.23], [0.46, 0.52])	([0.35, 0.38], [0.14, 0.19], [0.11, 0.29])

Table 2. Computed degree of unknown weights

	IVSFSWPA	IVSFSWPG	IVSFSWPWA	IVSFSWPWG
1	([0.3284, 0.4029], [0.3528, 0.4683], [0.3982, 0.4518])	([0.3226, 0.3982], [0.3561, 0.4729], [0.4151, 0.4706])	([0.3305, 0.4031], [0.3425, 0.4641], [0.3891, 0.4472])	([0.3238, 0.3978], [0.3453, 0.4684], [0.4083, 0.4697])
2	([0.2248, 0.5303], [0.2474, 0.3404], [0.2304, 0.3900])	([0.2231, 0.4669], [0.2561, 0.3448], [0.2387, 0.4393])	([0.2084, 0.3957], [0.2888, 0.3554], [0.2268, 0.3804])	([0.2073, 0.3575], [0.2976, 0.3611], [0.2347, 0.4283])
3	([0.4406, 0.5240], [0.2851, 0.3414], [0.1612, 0.2414])	([0.4381, 0.5200], [0.2919, 0.3542], [0.1619, 0.2432])	([0.4283, 0.5161], [0.2741, 0.3370], [0.1417, 0.2172])	([0.4265, 0.5125], [0.2823, 0.3523], [0.1424, 0.2184])
4	([0.1084, 0.1693], [0.5246, 0.6160], [0.2339, 0.2833])	([0.1083, 0.1692], [0.5453, 0.6311], [0.2403, 0.2903])	([0.1126, 0.1744], [0.5503, 0.6366], [0.1831, 0.2368])	([0.1125, 0.1743], [0.5727, 0.6531], [0.1869, 0.2408])
5	([0.3800, 0.4297], [0.3088, 0.4093], [0.3088, 0.4093])	([0.3737, 0.4218], [0.2913, 0.3457], [0.3180, 0.4206])	([0.4020, 0.4519], [0.2542, 0.3084], [0.2966, 0.3899])	([0.3980, 0.4467], [0.2610, 0.3156], [0.3063, 0.3998])

Step 2: Collected information having two types of attributes such as beneficial and non-beneficial. If given information contains more than one type of attribute, we need to transform the standard decision matrix into a normalized decision matrix; otherwise, this technique is necessary.

Step 3: Compute support by using the following expression, and the support degree between alternatives is given by:

$$\mathrm{supp}(\Delta_{ij}, \Delta_{jk}) = 1 - D(\Delta_{ij}, \Delta_{jk})$$

where the distance D between two alternatives is defined as:

$$D(\Delta_i, \Delta_\tau) = \frac{1}{6}\left(\left|(\mu_i^l)^2 - (\mu_k^l)^2\right| + |(\mu_j^u)^2 - (\mu_k^u)^2| + \left|(v_i^l)^2 - (v_j^l)^2\right| + |(v_j^u)^2 - (v_k^u)^2| + \left|(\pi_i^l)^2 - (\pi_j^l)^2\right| + |(\pi_j^u)^2 - (\pi_k^u)^2|\right)$$

Step 4: Calculate the degree of weighted support:

$$Æ(\Delta) = \sum_{i=1, i\neq\tau}^{n} E_i \,\mathrm{supp}(\Delta_i, \Delta_\tau)$$

where $E = (E_1, E_2, \ldots, E_n)$ and $E_i > 0$ and $\sum_{i=1,i\neq\tau}^{n} E_i = 1$.

Step 5: Investigate the degree of the support:

$$\Phi = \frac{E_i(1 + Æ(\Delta_i))}{\sum_{i=1,i\neq\tau}^{n} E_i(1 + Æ(\Delta_i))}$$

Step 6: The provided data is consolidated through the utilization of the IVSFSWPA, IVSFSWPG, IVSFSWPWA, and IVSFSWPWG operators as shown in Table 3.

Table 3. Covered score values corresponding to each alternative

	IVSFSWPA	**IVSFSWPG**	**IVSFSWPWA**	**IVSFSWPWG**
1	0.5212	0.5061	0.5292	0.5123
2	0.6498	0.6111	0.5981	0.5711
3	0.7289	0.7221	0.7313	0.7243
4	0.4169	0.4009	0.4151	0.3985
5	0.6231	0.6117	0.6554	0.6455

Step 7: In order to identify practical optimal choices, the score values of each individual are thoroughly examined in Table 4.

Table 4. Ranking of score values

Operator	**Ranking**
IVSFSWPA	$Æ_3 > Æ_2 > Æ_5 > Æ_1 > Æ_4$
IVSFSWPG	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
IVSFSWPWA	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
IVSFSWPWG	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$

Its cleary that $Æ_3$ is the best alternatives in all of them because $Æ_3$ has the higher score value in all different operators see in Figure 2.

4.3 Influence Study

This paragraph appears to describe a study or analysis where the parameter ϑ is varied from 0 to 100, and the behavior of the ranking orders of alternatives is observed. Despite the variation in the parameter value, the ranking orders of the alternatives remain the same throughout the range of values tested (from 0 to 100). This suggests that the ranking orders of the alternatives are stable and not influenced by changes in the parameter ϑ within this range in Tables 5 and 6. Its cleary that $Æ_3$ is the best alternatives in all of them because $Æ_3$ has the higher score value in all different operators. So, we find the result that Rainwater Harvesting $Æ_3$ is best function of Recycled water.

4.4 Comparative Analysis

In this section we use the data in Table 7 to compare a group of current employees with previous employees. Using various examples, we managed to show a significant improvement in the performance of existing employees by comparing specific skills and the current situation to increase the cost and power of medical personnel. Sarkar et al. [30] presented an application of Sugeno-Weber TN regarding collective workers. Hadzic et al. [37]showed the application of fixed point theory in the field of probability measurement. Ashraf et al. [39] gave the concept of Sugeno weber aggregation operators using the adaption programming for media platform.

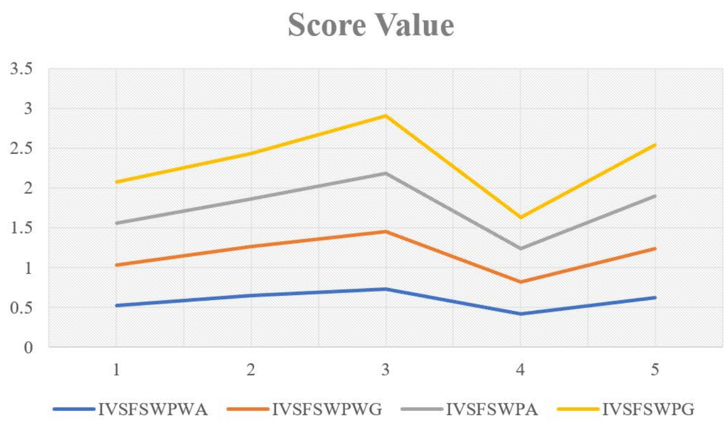

Figure 2. Results of alternatives

Table 5. Results obtained by IVSFSWPWA

Parametric Value	Ranking and Ordering
$\vartheta = 1$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 5$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 20$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 35$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 55$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 70$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 80$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 90$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 100$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$

Table 6. Results obtained by IVSFSWPWG

Parametric Value	Ranking and Ordering
$\vartheta = 1$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 5$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 20$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 35$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 55$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 70$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 80$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 90$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
$\vartheta = 100$	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$

Table 7. Analysis of comparative study

Methods	Ranking Information
IVSFSWPA	$Æ_3 > Æ_2 > Æ_5 > Æ_1 > Æ_4$
IVSFSWPG	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
IVSFSWPWA	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
IVSFSWPWG	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Sarkar et al. [30]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Sarkar et al. [30]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Hadzic et al. [37]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Hadzic et al. [37]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Ashraf et al. [39]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$
Ashraf et al. [39]	$Æ_3 > Æ_5 > Æ_2 > Æ_1 > Æ_4$

5 Conclusions

In this article, we first introduced the fundamental operations in IVSFS using the Sugeno-Weber TN and TCN. Using the Sugeno-Weber AOs, IVSFS data which comprised four distinct types of AOs (IVSFWPWA and IVSFWPWG operators was then combined. The paper highlights some of these AOs' notable properties, including boundedness, idempotency, and monotonicity. Application scenarios of IVSFWPWA and IVSFWPWG operators were demonstrated in solving MADM problems through the use of a real-world example involving the prioritization and assessment of business projects completed by a construction company. The behavior of these operators was investigated further by changing the value of the involved parameter Δ. With the use of an experimental case study that took into account significant variables or characteristics, the right recycled water was chosen. The effectiveness of the derived approaches was validated by a comparative analysis that compared the outcomes with those of existing approaches and developed aggregation techniques. These approaches were shown to be applicable in the impact study. Our main focus in the future will be on theoretical and applied research on T-Spherical Fuzzy, Complex Bipolar Soft Sets, and Complex Spherical Fuzzy.

Author Contributions

Conceptualization, M.S.; writing-review, M.S.; editing, D.B.; validation, D.B.; supervision, D.B.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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