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Linear Diophantine Fuzzy Graph Structures: Theoretical Foundations and Applications in Road Crime Detection

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Abstract: Graph structures (GSs) have appeared as a robust mathematical framework for modelling and resolving complex combinatorial problems across diverse realms. At the same time, the linear Diophantine fuzzy set (LDFS) is a noteworthy expansion of the conventional concepts of the fuzzy set (FS), intuitionistic fuzzy set (IFS), Pythagorean fuzzy set (PFS), and q-Rung orthopair fuzzy set (q-ROFS). The LDFS framework introduces a flexible parameterization strategy that independently relaxes membership and non-membership restraints through reference parameters, thereby attaining enhanced expressiveness in apprehending ambiguous real-world phenomena. In this paper, a novel concept of linear Diophantine fuzzy graph structure (LDFGS) is introduced as a generalization of intuitionistic fuzzy graph structure (IFGS) and linear Diophantine fuzzy graph (LDFG) to GSs. Several cardinal fundamental notions in LDFGSs, including $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and the total degree of a vertex in an LDFGS are discussed. Additionally, $\breve{\rho}_i$-size of an LDFGS, the size of an LDFGS, and the order of an LDFGS are studied. Meanwhile, the ideas of the maximal product of two LDFGSs, strong LDFGS, degree, and $\breve{\rho}_i$-degree of the maximal product are introduced with several concrete illustrations. To empirically validate the efficacy and practical utility of the proposed LDFGS framework, this study presents a case study analyzing road crime patterns across heterogeneous urban regions in Sindh province, Pakistan.

Keywords: Graph structure (GS); Linear Diophantine fuzzy set (LDFS); Maximal product; Degree of a vertex; Detection of road crimes

1 Introduction

The theory of FSs was proposed by Zadeh [1] to model uncertainty by assigning a membership degree (MD) to each element within a set. However, in many real-world scenarios, MD alone is insufficient to capture the full spectrum of uncertainty. To address this limitation, Atanassov [2] proposed the IFS, which incorporates both MD and NMD under the constraint that their sum does not exceed unity. Despite the advancements offered by IFS, several practical dilemmas require a more generalized framework where the IFS condition is not always satisfied. In response to this need, Yager [3] introduced the notion of PFS, which relaxes the IFS condition by ensuring that the sum of the squared MD and NMD does not exceed one. Subsequently, Yager [4] further extended this concept by proposing q-ROFS, wherein the total of the q-th power of the MD and NMD remains within the unit interval. This generalization enhances data representation capabilities, enabling a more refined characterization of ambiguity and imprecise information. While the membership and non-membership degrees in IFS, PFS, and q-ROFS are constrained by specific mathematical conditions, these restrictions limit their flexibility in modelling uncertainty. To address these constraints, Riaz and Hashmi [5] introduced a generalization variant of FS known as LDFS. By incorporating reference parameters, LDFSs provide a larger and more adaptable framework for defining MD and NMD, surpassing the restrictions inherent in previous models. This enhanced flexibility makes LDFSs a more robust and reliable approach for managing uncertainty. Given the progress and the greater degree of liberty offered by LDFSs, this framework has attracted substantial scholarly attention. Academics have begun exploring and expanding upon this novel concept, leading to the development of new theoretical extensions and applications [6–9]. Kamacı [10] studied linear Diophantine fuzzy algebraic structures.

Graph theory (GT) was invented from Euler's solution to the Königsberg bridge problem, which required to determining a path that traversed each of the city's seven bridges exactly once. Since then, GT has evolved into a vital discipline with applications spanning various fields of science and the humanities. It is an influential instrument for modelling and solving complicated realistic issues, mainly in network representation. Networks modelled using GT involve a wide range of applications, including city transportation systems, telephone networks, recommendation algorithms, computer and circuit networks, and social media platforms such as LinkedIn and Facebook. For instance, in Facebook's social network, each user is represented as a vertex (or node), with edges indicating connections between users. Each node stores attributes such as name, ID, gender, and location, facilitating efficient data organization and retrieval. Beyond social networks, GT plays a crucial role in sports analytics, where it aids in analyzing player interactions and team dynamics. A football match, for instance, can be modelled as a graph, with players represented as nodes and their on-field interactions as edges. Additionally, GT is instrumental in solving spatial and structural problems. For example, it can be applied to navigation challenges, where an agent must traverse a network of interconnected locations while ensuring optimal movement. Likewise, in autonomous vehicle coordination, GT can model the placement and movement of cars at intersections to optimize traffic flow and minimize redundancies.

A graph is a mathematical representation of networks, illustrating relationships between vertices and edges. In such a model, vertices may represent entities such as workstations, while edges denote connections between them. However, conventional graphs often fail to accurately capture many real-world processes due to the inherent complexity and variability of structural features. To address these limitations, Rosenfeld [11] introduced the concept of fuzzy graphs (FGs). Fuzzy GT (FGT) has numerous fields, such as data mining, networking, clustering, planning, image capturing, and scheduling. More literature on FGs can be seen in the study [12]. Parvathi et al. [13] employed IFS to study intuitionistic fuzzy graphs (IFGs) and their fundamental operations. Akram et al. [14] and Akram et al. [15] framed certain PF graphs (PFGs) and q-ROF graphs (q-ROFGs). Hanif et al. [16] deliberate the idea of a linear Diophantine fuzzy graph (LDFG).

A graph is a pair of sets of vertices $\mathbb{V}$ and a relation $\mathbb{E}$ on $\mathbb{V}$, which is capable of describing a wide range of real-world events. Though, in many real dilemmas that concern more than one relation, GT cannot work efficiently. To address such circumstances, Sampathkumar [17] generalized the idea of graphs and constructed the theory of Graph structures (GSs). GS has 'n' mutually disjoint, symmetric, and irreflexive relations. Following that, Akram and Sitara [18] and Sitara and Akram [19] examined the degree, total degree, and a few properties of the semi-strong min product, maximal product, and residue product of fuzzy GSs (FGSs). Sharma and Bansal [20] introduced the concept of intuitionistic fuzzy graph structure (IFGS). The authors studied the concept of the q-rung picture fuzzy graph structure (q-RPFGS) [21].

1.1 Knowledge Gaps and Motivations

The main motivations and research gaps behind this paper are outlined as follows:

- GSs serve as an important tool in advancing research across numerous domains of computer science and computational intelligence. FGSs offer a noteworthy advantage over traditional GSs, as they successfully handle the uncertainty and ambiguity innate in real-world dilemmas.
- The most recent development in FS theory, known as the LDFS framework, was familiarized by Riaz and Hashmi [5] to address the restrictions allied with MD and NMD in earlier approaches such as FS, IFS, PFS, and q-ROFS. By integrating reference parameters, LDFS provides decision-makers with better flexibility in their assessments, enhancing the decision-making scenarios.
- Hanif et al. [16] recently presented the notion of LDFGs and established their essential operations and properties. LDFGs offer significant advantages over FGs, IFGs, PFGs, and q-ROFGs, as they incorporate a broader range of MD and NMD, thereby enhancing their applicability in complicated decision-making scenarios.
- An analysis of the existing literature indicates a noticeable lack of research on linear Diophantine fuzzy graph structures (LDFGSs). While various extensions of FGs have been explored, the integration of graph structures within the framework of LDFSs and their potential application remain unexplored. Given the enhanced flexibility and broader representational abilities of LDFSs, their application to graph theory holds significant potential for addressing complex real-world problems involving uncertainty. To bridge this knowledge gap, we introduce and formalize the idea of LDFGS by extending the notion of graph structures within the LDFS context.

1.2 Main Contributions

Based on the above discussion, the main contributions of this study are listed as follows:

- To establish a detailed study on GS in the context of LDFSs and introduce the concept of LDFGS.
- To develop the idea of the maximal product of two LDFGSs.
- To explore the concept of strong LDFGS and examine the degree and $\breve{\rho}_i$-degree of the maximal product.

- Furthermore, to show its efficacy and influence in real-life paradigms, an application of LDFGS regarding the identification of road crime detection among cities in the province of Sindh is presented.

1.3 Framework of This Work

To promote our discussion, this article is organized as follows: Section 2 offers a review of the rudimentary information necessary for the subsequent discussions. Section 3 introduces the idea of LDFGSs, along with essential definitions and illustrative examples. Section 4 explores the maximal product of two LDFGSs, introduces the notion of strong LDFGS, and examines the degree and $\breve{\rho}_i$-degree of the maximal product. Furthermore, to show its efficacy and influence in real-life problems in Section 6 an application of LDFGS regarding the identification of road crime detection among cities in the province of Sindh. Section 7 presents some concluding remarks.

2 Basic Concepts

In this segment, some essential ideas of IFS, LDFS, IFR, LDFR, GS, FGS and IFGS. Throughout this paper, unless stated otherwise, $\mathbb{V}$, $\mathbb{V}_1$, and $\mathbb{V}_2$ denote universal sets.

Definition1. An IFS $\mathcal{I}$ over $\mathbb{V}$ is portrayed as [2]:

$$\mathcal{I} = \Big\{ \Big(\flat, \big\langle \varkappa^m(\flat), \varkappa^n(\flat) \big\rangle \Big) : \flat \in \mathbb{V} \Big\}, \tag{1}$$

where, $\varkappa^m, \varkappa^n : \mathbb{V} \longrightarrow [0,1]$ represents the MD and NMD, respectively, such that $0 \leq \varkappa^m(\flat) + \varkappa^n(\flat) \leq 1, \forall \flat \in \mathbb{V}$.

Definition2. An LDFS $\mathfrak{L}$ over $\mathbb{V}$ is postulated as [5]:

$$\mathfrak{L} = \Big\{ \Big(\flat, \big\langle \varkappa^m(\flat), \varkappa^n(\flat) \big\rangle, \big\langle \alpha, \beta \big\rangle \Big) : \flat \in \mathbb{V} \Big\}, \tag{2}$$

where, $\varkappa^m, \varkappa^n : \mathbb{V} \longrightarrow [0,1]$ are MD and NMD, and $\alpha, \beta \in [0,1]$ are corresponding reference parameters, respectively, with $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha\varkappa^m(\flat) + \beta\varkappa^n(\flat) \leq 1, \forall\, \flat \in \mathbb{V}$. The degree of hesitation of any $\flat \in \mathbb{V}$ is articulated as $\pi(\flat) = 1 - \big(\alpha\varkappa^m(\flat) + \beta\varkappa^n(\flat)\big)$.

We will use $\mathfrak{LDFS}(\mathbb{V})$ for the collection of all LDFSs over $\mathbb{V}$. For convenience, we will use $\mathfrak{L} = (\langle \varkappa^m, \varkappa^n \rangle, \langle \alpha, \beta \rangle)$ for an LDFS over $\mathbb{V}$.

Definition3. Let $\mathfrak{L}_1 = \Big(\big\langle \varkappa_1^m, \varkappa_1^n \big\rangle, \big\langle \alpha_1, \beta_1 \big\rangle \Big)$ and $\mathfrak{L}_2 = \Big(\big\langle \varkappa_2^m, \varkappa_2^n \big\rangle, \big\langle \alpha_2, \beta_2 \big\rangle \Big)$ be two LDFSs on $\mathbb{V}$. Then, $\forall$ $\flat \in \mathbb{V}$ [5]

(1) $\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \Leftrightarrow \varkappa_1^m \leq \varkappa_2^m, \varkappa_1^n \geq \varkappa_2^n$, and $\alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2$;

(2) $\mathfrak{L}_1 \cup \mathfrak{L}_2 = \Big(\big\langle \varkappa_1^m \vee \varkappa_2^m, \varkappa_1^n \wedge \varkappa_2^n \big\rangle, \big\langle \alpha_1 \vee \alpha_2, \beta_1 \wedge \beta_2 \big\rangle \Big)$;

(3) $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \Big(\big\langle \varkappa_1^m \wedge \varkappa_2^m, \varkappa_1^n \vee \varkappa_2^n \big\rangle, \big\langle \alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2 \big\rangle \Big)$;

(4) $\mathfrak{L}_1^c = \Big(\big\langle \varkappa_1^n, \varkappa_1^m \big\rangle, \big\langle \beta_1, \alpha_1 \big\rangle \Big)$;

Definition4. An IFR ϱ from $\mathbb{V}_1$ to $\mathbb{V}_2$ is described as [22]:

$$\varrho = \Big\{ \Big((\flat_1, \flat_2), \big\langle \varkappa_\varrho^m(\flat_1, \flat_2), \varkappa_\varrho^n(\flat_1, \flat_2) \big\rangle \Big) : \flat_1 \in \mathbb{V}_1, \flat_2 \in \mathbb{V}_2 \Big\}, \tag{3}$$

where, $\varkappa_\varrho^m, \varkappa_\varrho^n : \mathbb{V}_1 \times \mathbb{V}_2 \longrightarrow [0,1]$ express the MD and NMD from $\mathbb{V}_1$ to $\mathbb{V}_2$, respectively such that $0 \leq \varkappa_\varrho^m(\flat_1, \flat_2)$ $+ \varkappa_\varrho^n(\flat_1, \flat_2) \leq 1, \forall\, (\flat_1, \flat_2) \in \mathbb{V}_1 \times \mathbb{V}_2$.

Collection of all IFRs from $\mathbb{V}_1$ to $\mathbb{V}_2$ will be symbolized by $\mathfrak{IFS}(\mathbb{V}_1 \times \mathbb{V}_2)$.

Definition5. Presume that $\varrho_1 = \big\langle \varkappa_{\varrho_1}^m(\flat_1, \flat_2), \varkappa_{\varrho_1}^n(\flat_1, \flat_2) \big\rangle$ be an IFR from $\mathbb{V}_1$ to $\mathbb{V}_2$ and $\varrho_2 = \big\langle \varkappa_{\varrho_2}^m(\flat_1, \flat_2),$ $\varkappa_{\varrho_2}^n(\flat_1, \flat_2) \big\rangle$ be an IFR from $\mathbb{V}_2$ to $\mathbb{V}_3$. Then, their composition $\varrho_1 \circ \varrho_2 \in \mathfrak{IFS}(\mathbb{V}_1 \times \mathbb{V}_2)$ is postulated as [22]:

$$\varrho_1 \circ \varrho_2 = \Big\langle \big(\varkappa_{\varrho_1}^m \circ \varkappa_{\varrho_2}^m\big)(\flat_1, \flat_3), \big(\varkappa_{\varrho_1}^n \circ \varkappa_{\varrho_2}^n\big)(\flat_1, \flat_3) \Big\rangle, \tag{4}$$

where,

$$\big(\varkappa_{\varrho_1}^m \circ \varkappa_{\varrho_2}^m\big)(\flat_1, \flat_3) = \bigvee_{\flat_2 \in \mathbb{V}_2} \Big(\varkappa_{\varrho_1}^m(\flat_1, \flat_2) \wedge \varkappa_{\varrho_2}^m(\flat_2, \flat_3) \Big), \tag{5}$$

$$\big(\varkappa_{\varrho_1}^n \circ \varkappa_{\varrho_2}^n\big)(\flat_1, \flat_3) = \bigwedge_{\flat_2 \in \mathbb{V}_2} \Big(\varkappa_{\varrho_1}^n(\flat_1, \flat_2) \vee \varkappa_{\varrho_2}^n(\flat_2, \flat_3) \Big), \tag{6}$$

$$\forall\, \flat_1 \in \mathbb{V}_1, \flat_2 \in \mathbb{V}_2, \flat_3 \in \mathbb{V}_3.$$

Definition6. Let $\varrho = \big\langle \varkappa_{\varrho}^{m}(\flat_1, \flat_2), \varkappa_{\varrho}^{n}(\flat_1, \flat_2)\big\rangle$ be an IFR from $\mathbb{V}_1$ to $\mathbb{V}_2$. Then,

$$\mathfrak{supp}(\varrho) = \Big\{(\flat_1, \flat_2) : \varkappa_{\varrho}^{m}(\flat_1, \flat_2) > 0, \varkappa_{\varrho}^{n}(\flat_1, \flat_2) > 0\Big\}, \tag{7}$$

is titled as the support of ϱ.

Definition7. An LDFR $\tilde{\rho}$ from $\mathbb{V}_1$ to $\mathbb{V}_2$ is articulated as [6]:

$$\tilde{\rho} = \Big\{\Big((\flat_1, \flat_2), \big\langle \varkappa_{\tilde{\rho}}^{m}(\flat_1, \flat_2), \varkappa_{\tilde{\rho}}^{n}(\flat_1, \flat_2)\big\rangle, \big\langle \alpha_{\tilde{\rho}}(\flat_1, \flat_2), \beta_{\tilde{\rho}}(\flat_1, \flat_2)\big\rangle\Big) : \flat_1 \in \mathbb{V}_1, \flat_2 \in \mathbb{V}_2\Big\}, \tag{8}$$

where, $\varkappa_{\tilde{\rho}}^{m}, \varkappa_{\tilde{\rho}}^{n} : \mathbb{V}_1 \times \mathbb{V}_2 \longrightarrow [0, 1]$ represent the MD and NMD from $\mathbb{V}_1$ to $\mathbb{V}_2$, and $\alpha_{\tilde{\rho}}(\flat_1, \flat_2), \beta_{\tilde{\rho}}(\flat_1, \flat_2) \in [0, 1]$ are the corresponding reference parameters, respectively subjected to the constraint $0 \leq \alpha_{\tilde{\rho}}(\flat_1, \flat_2)\varkappa_{\tilde{\rho}}^{m}(\flat_1, \flat_2) + \beta_{\tilde{\rho}}(\flat_1, \flat_2)\varkappa_{\tilde{\rho}}^{n}(\flat_1, \flat_2) \leq 1 \;\forall\; (\flat_1, \flat_2) \in \mathbb{V}_1 \times \mathbb{V}_2$ with $0 \leq \alpha_{\tilde{\rho}}(\flat_1, \flat_2) + \beta_{\tilde{\rho}}(\flat_1, \flat_2) \leq 1$. The hesitation degree can be evaluated as:

$$\pi(\flat_1, \flat_2) = 1 - \Big(\alpha_{\tilde{\rho}}(\flat_1, \flat_2)\varkappa_{\tilde{\rho}}^{m}(\flat_1, \flat_2) + \beta_{\tilde{\rho}}(\flat_1, \flat_2)\varkappa_{\tilde{\rho}}^{n}(\flat_1, \flat_2)\Big). \tag{9}$$

The assemblage of all LDFRs from $\mathbb{V}_1$ to $\mathbb{V}_2$ by $\mathfrak{LDFS}(\mathbb{V}_1 \times \mathbb{V}_2)$.

Definition8. Let $\tilde{\rho_1} = \Big(\big\langle \varkappa_{\tilde{\rho_1}}^{m}(\flat_1, \flat_2), \varkappa_{\tilde{\rho_1}}^{n}(\flat_1, \flat_2)\big\rangle, \big\langle \alpha_{\tilde{\rho_1}}(\flat_1, \flat_2), \beta_{\tilde{\rho_1}}(\flat_1, \flat_2)\big\rangle\Big)$ be an LDFR from $\mathbb{V}_1$ to $\mathbb{V}_2$ and $\tilde{\rho_2} = \Big(\big\langle \varkappa_{\tilde{\rho_2}}^{m}(\flat_2, \flat_3), \varkappa_{\tilde{\rho_2}}^{n}(\flat_2, \flat_3)\big\rangle, \big\langle \alpha_{\tilde{\rho_2}}(\flat_2, \flat_3), \beta_{\tilde{\rho_2}}(\flat_2, \flat_3)\big\rangle\Big)$ be an LDFR from $\mathbb{V}_2$ to $\mathbb{V}_3$. Then, their composition is described as [6]:

$$\tilde{\rho_1} \circ \tilde{\rho_2} = \Big(\big\langle \big(\varkappa_{\tilde{\rho_1}}^{m} \circ \varkappa_{\tilde{\rho_2}}^{m}\big)(\flat_1, \flat_3), \big(\varkappa_{\tilde{\rho_1}}^{n} \circ \varkappa_{\tilde{\rho_2}}^{n}\big)(\flat_1, \flat_3)\big\rangle, \big\langle \big(\alpha_{\tilde{\rho_1}} \circ \alpha_{\tilde{\rho_2}}\big)(\flat_1, \flat_3), \big(\beta_{\tilde{\rho_1}} \circ \beta_{\tilde{\rho_2}}\big)(\flat_1, \flat_3)\big\rangle\Big) \tag{10}$$

where,

$$\big(\varkappa_{\tilde{\rho_1}}^{m} \circ \varkappa_{\tilde{\rho_2}}^{m}\big)(\flat_1, \flat_3) = \bigvee_{\flat_2 \in \mathbb{V}_2} \Big(\varkappa_{\tilde{\rho_1}}^{m}(\flat_1, \flat_2) \wedge \varkappa_{\tilde{\rho_2}}^{m}(\flat_2, \flat_3)\Big), \tag{11}$$

$$\big(\varkappa_{\tilde{\rho_1}}^{n} \circ \varkappa_{\tilde{\rho_2}}^{n}\big)(\flat_1, \flat_3) = \bigwedge_{\flat_2 \in \mathbb{V}_2} \Big(\varkappa_{\tilde{\rho_1}}^{n}(\flat_1, \flat_2) \vee \varkappa_{\tilde{\rho_2}}^{n}(\flat_2, \flat_3)\Big), \tag{12}$$

$$\big(\alpha_{\tilde{\rho_1}} \circ \alpha_{\tilde{\rho_2}}\big)(\flat_1, \flat_3) = \bigvee_{\flat_2 \in \mathbb{V}_2} \Big(\alpha_{\tilde{\rho_1}}(\flat_1, \flat_2) \wedge \alpha_{\tilde{\rho_2}}(\flat_2, \flat_3)\Big), \tag{13}$$

$$\big(\beta_{\tilde{\rho_1}} \circ \beta_{\tilde{\rho_2}}\big)(\flat_1, \flat_3) = \bigwedge_{\flat_2 \in \mathbb{V}_2} \Big(\beta_{\tilde{\rho_1}}(\flat_1, \flat_2) \vee \beta_{\tilde{\rho_2}}(\flat_2, \flat_3)\Big), \tag{14}$$

$$\forall\; \flat_1 \in \mathbb{V}_1, \flat_2 \in \mathbb{V}_2, \flat_3 \in \mathbb{V}_3.$$

Definition9. Let $\tilde{\rho} = \Big(\big\langle \varkappa_{\tilde{\rho}}^{m}(\flat_1, \flat_2), \varkappa_{\tilde{\rho}}^{n}(\flat_1, \flat_2)\big\rangle, \big\langle \alpha_{\tilde{\rho}}(\flat_1, \flat_2), \beta_{\tilde{\rho}}(\flat_1, \flat_2)\big\rangle\Big)$ be an LDFR from $\mathbb{V}_1$ to $\mathbb{V}_2$. Then, the set

$$\mathfrak{supp}(\tilde{\rho}) = \big\{(\flat_1, \flat_2) : \varkappa_{\tilde{\rho}}^{m}(\flat_1, \flat_2) > 0, \varkappa_{\tilde{\rho}}^{n}(\flat_1, \flat_2) > 0\rangle, \langle \alpha_{\tilde{\rho}}(\flat_1, \flat_2) > 0, \beta_{\tilde{\rho}}(\flat_1, \flat_2) > 0\big\} \tag{15}$$

is named the support of $\tilde{\rho}$.

2.1 Fuzzy Graph Structures

Definition10. Let $\mathbb{V}$ be any non void set known as the vertex set and $\mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_k$ be mutually disjoint relations (sets of edges) of $\mathbb{V}$ such that each $\mathbb{E}_i$, $1 \leq i \leq k$ is symmetric and irreflexive. Then, $\mathscr{G} = \big(\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_k\big)$ is named a graph structure (GS) [17].

Definition11. Let $\mathscr{G} = \big(\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_k\big)$ be a GS. Then, $\hat{\mathscr{G}} = \big(\mathscr{F}, \rho_1, \rho_2, \cdots, \rho_k\big)$ is called fuzzy graph structure (FGS) of GS $\mathscr{G}$, where $\mathscr{F} \in \mathbf{FS}(\mathbb{V})$ and $\rho_i \in \mathbf{FS}(\mathbb{E}_i)$, $\forall\; 1 \leq i \leq k$, if $0 \leq \varkappa_{\rho_i}^{m}(\mathbf{x}, \mathbf{y}) \leq \varkappa_{\mathscr{F}}^{m}(\mathbf{x}) \wedge \varkappa_{\mathscr{F}}^{m}(\mathbf{y})$ $\forall\; \mathbf{x}, \mathbf{y} \in \mathbb{V}$, $i = 1, 2, \cdots, k$.

From now onward, we will simply utilize $\mathscr{G}$ for a GS.

Definition12. Let $\hat{\mathscr{G}} = \big(\mathscr{F}, \rho_1, \rho_2, \cdots, \rho_k\big)$ be a FGS with GS $\mathscr{G}$ [23].

(1) If $(\mathbf{x}, \mathbf{y}) \in \mathfrak{supp}(\rho_i)$, then $(\mathbf{x}, \mathbf{y})$ is named ρ_i-edge of $\hat{\mathscr{G}}$.

(2) A ρ_i-path of $\hat{\mathscr{G}}$ is a sequence of vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r$ which are distinct except possibly $\mathbf{x}_0 = \mathbf{x}_r$, such that $(\mathbf{x}_{j-1}, \mathbf{x}_j)$ is a ρ_i-edge $\forall\; j = 1, 2, \cdots, r$.

(3) Two vertices of $\hat{\mathscr{G}}$ are said to be ρ_i-connected if they are joined by ρ_i-path.

(4) The strength of ρ_i-path $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r$ of a FGS $\hat{\mathscr{G}}$ is $\bigwedge_{j=1}^{r} \varkappa_{\rho_i}^m(\mathbf{x}_{j-1}, \mathbf{x}_j)$ for $i = 1, 2, \cdots, k$.

(5) In FGS $\hat{\mathscr{G}}$, $(\varkappa_{\rho_i}^m)^\infty(\mathbf{x}, \mathbf{y}) = \vee\{(\varkappa_{\rho_i}^m)^j(\mathbf{x}, \mathbf{y}) : j = 2, 3, \cdots, k\}$, where $(\varkappa_{\rho_i}^m)^j = ((\varkappa_{\rho_i}^m)^{j-1} \circ \varkappa_{\rho_i}^m)(\mathbf{x}, \mathbf{y})$, $j \geq 2$.

(6) The ρ_i-degree of a vertex $\mathbf{x}$ in $\hat{\mathscr{G}}$ is defined as $\mathbb{D}_{\rho_i}(\mathbf{x}) = \sum_{\mathbf{x} \neq \mathbf{y}, (\mathbf{x}, \mathbf{y}) \in \mathbb{E}_i} \varkappa_{\rho_i}^m(\mathbf{x}, \mathbf{y})$.

(7) The degree of the vertex $\mathbf{x} \in \mathbb{V}$ is denoted and defined by $\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{D}_{\rho_i}(\mathbf{x})$.

(8) $\hat{\mathscr{G}}$ is said to be $\mathbf{a}$-ρ_i regular, if $\mathbb{D}_{\rho_i}(\mathbf{x}) = \mathbf{a}, \forall\, \mathbf{x} \in \mathbb{V}$. Moreover, $\hat{\mathscr{G}}$ is called $\mathbf{a}$-regular, if $\mathbb{D}(\mathbf{x}) = \mathbf{a}, \forall\, \mathbf{x} \in \mathbb{V}$.

Definition13. Let $\hat{\mathscr{G}}_1 = (\mathscr{F}_1, \rho_1', \rho_2', ..., \rho_k')$ and $\hat{\mathscr{G}}_2 = (\mathscr{F}_2, \rho_1'', \rho_2'', ..., \rho_k'')$ be two FGSs with underlying GSs $\mathscr{G}_1 = (\mathbb{V}_1, \mathbb{E}_1', \mathbb{E}_2', ..., \mathbb{E}_k')$ and $\mathscr{G}_2 = (\mathbb{V}_2, \mathbb{E}_1'', \mathbb{E}_2'', ..., \mathbb{E}_k'')$, respectively. Then, $\hat{\mathscr{G}} = \hat{\mathscr{G}}_1 * \hat{\mathscr{G}}_2 = (\mathscr{F}, \rho_1, \rho_2, ..., \rho_k)$ is called maximal FGS with underlying crisp GS $G = (\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_k)$, where $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ and $\mathbb{E}_i = \{(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2) : \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}_i \text{ or } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}_i'\}$. Fuzzy vertex set $\mathscr{F}$ and fuzzy relations ρ_i in maximal product $\hat{\mathscr{G}}$ are defined as [19]:

$$\mathscr{F} = \mathscr{F}_1 * \mathscr{F}_2,\ \varkappa_{\mathscr{F}}^m(\mathbf{x}, \mathbf{y}) = \varkappa_{\mathscr{F}_1}^m(\mathbf{x}) \vee \varkappa_{\mathscr{F}_2}^m(\mathbf{y}),\ \forall\,(\mathbf{x}, \mathbf{y}) \in \mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2,\ and\ \rho_i = \rho_i' * \rho_i'',$$

$$\varkappa_{\rho_i}^m(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2) = \begin{cases} \varkappa_{\mathscr{F}_1}^m(\mathbf{x}_1) \vee \varkappa_{\rho_i''}^m(\mathbf{y}_1, \mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}_i'' \\ \varkappa_{\mathscr{F}_2}^m(\mathbf{y}_1) \vee \varkappa_{\rho_i'}^m(\mathbf{x}_1, \mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}_i' \end{cases}$$

$$i = 1, 2, ..., k.$$

Definition14. A FGS $\hat{\mathscr{G}} = (\mathscr{F}, \rho_1, \rho_2, ..., \rho_n)$ is ρ_i-strong, if [19]

$$\varkappa_{\rho_i}^m(\mathbf{x}, \mathbf{y}) = \varkappa_{\mathscr{F}}^m(\mathbf{x}) \wedge \varkappa_{\mathscr{F}}^m(\mathbf{y}),\ \forall\,(\mathbf{x}, \mathbf{y}) \in \mathbb{E}_i, i \in \{1, 2, ..., k\}.$$

If $\hat{\mathscr{G}}$ is ρ_i-strong $\forall\, i \in \{1, 2, ..., k\}$, then $\hat{\mathscr{G}}$ is called strong FGS.

Definition15. The degree of a vertex in maximal product $\hat{\mathscr{G}} = \hat{\mathscr{G}}_1 * \hat{\mathscr{G}}_2$ of two FGSs $\hat{\mathscr{G}}_1 = (\mathscr{F}_1, \rho_1', \rho_2', ..., \rho_k')$ and $\hat{\mathscr{G}}_2 = (\mathscr{F}_2, \rho_1'', \rho_2'', ..., \rho_k'')$ is given by [19]:

$$\mathbb{D}_{\hat{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \sum_{(\mathbf{x}_i, \mathbf{x}_k) \in \mathbb{E}_i', \mathbf{y}_j = \mathbf{y}_l} \varkappa_{\rho_i'}^m(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa_{F_2}^m(\mathbf{y}_j) + \sum_{(\mathbf{y}_j, \mathbf{y}_l) \in \mathbb{E}_i'', \mathbf{x}_i = \mathbf{x}_k} \varkappa_{\rho_j''}^m(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa_{\mathscr{F}_1}^m(\mathbf{x}_i). \tag{16}$$

ρ_i-degree of a vertex of maximal product $\hat{\mathscr{G}}$ is postulated by:

$$\rho_i - \mathbb{D}_{\hat{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \sum_{(\mathbf{x}_i, \mathbf{x}_k) \in \mathbb{E}_i', \mathbf{y}_j = \mathbf{y}_l} \varkappa_{\rho_i'}^m(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa_{\mathscr{F}_2}^m(\mathbf{y}_j) + \sum_{(\mathbf{y}_j, \mathbf{y}_l) \in \mathbb{E}_i'', \mathbf{x}_i = \mathbf{x}_k} \varkappa_{\rho_j''}^m(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa_{\mathscr{F}_1}^m(\mathbf{x}_i). \tag{17}$$

2.2 Intuitionistic Fuzzy Graph Structures (IFGS)

Definition16. Let $\mathscr{G} = (\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_n)$ be a GS, $\mathcal{I} = \langle \varkappa_{\mathcal{I}}^m(\mathbf{x}), \varkappa_{\mathcal{I}}^n(\mathbf{x}) \rangle$ be an IFS on $\mathbb{V}$ and $\dot{\rho}_i = \langle \varkappa_{\dot{\rho}_i}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\dot{\rho}_i}^n(\mathbf{x}_1, \mathbf{x}_2) \rangle$ be irreflexive, symmetric and mutually disjoint IFRs from $\mathbb{V}_1$ to $\mathbb{V}_2$, $i = 1, 2, ..., n$, where $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$. Then, $\dot{\mathscr{G}} = (\mathcal{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_n)$ is called intuitionistic fuzzy graph structure (IFGS) of GS $\mathscr{G}$, if [20]

$$\varkappa_{\dot{\rho}_i}^m(\mathbf{x}_1, \mathbf{x}_2) \leq \varkappa_{\mathcal{I}}^m(\mathbf{x}) \wedge \varkappa_{\mathcal{I}}^n(\mathbf{x}_2),\ \text{and}\ \varkappa_{\dot{\rho}_i}^n(\mathbf{x}_1, \mathbf{x}_2) \geq \varkappa_{\mathcal{I}}^n(\mathbf{x}_1) \vee \varkappa_{\mathcal{I}}^n(\mathbf{x}_2),\ \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}, i = 1, 2, ..., n.$$

Definition17. Let $\dot{G} = (\mathcal{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_n)$ be an IFGS with the underlying GS $\mathscr{G} = (\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_n)$. Then, $\dot{\rho}_i$-strength of connectedness between any two vertices $\mathbf{x}_1, \mathbf{x}_2$ is defined by $(\dot{\rho}_i)^\infty(\mathbf{x}_1, \mathbf{x}_2) = \langle (\varkappa_{\dot{\rho}_i}^m)^\infty(\mathbf{x}_1, \mathbf{x}_2), (\varkappa_{\dot{\rho}_i}^n)^\infty(\mathbf{x}_1, \mathbf{x}_2) \rangle$ [20],
where,

$$(\varkappa_{\dot{\rho}_i}^m)^\infty(\mathbf{x}_1, \mathbf{x}_2) = \bigvee_{j=1}^{\infty} (\varkappa_{\dot{\rho}_i}^m)^j(\mathbf{x}_1, \mathbf{x}_2),\ \text{and}\ (\varkappa_{\dot{\rho}_i}^n)^\infty(\mathbf{x}_1, \mathbf{x}_2) = \bigwedge_{j=1}^{\infty} (\varkappa_{\dot{\rho}_i}^n)^j(\mathbf{x}_1, \mathbf{x}_2).$$

where,

$$(\varkappa_{\dot{\rho}_i}^m)^j(\mathbf{x}_1, \mathbf{x}_2) = ((\varkappa_{\dot{\rho}_i}^m)^{j-1} \circ \varkappa_{\dot{\rho}_i}^m)(\mathbf{x}_1, \mathbf{x}_2) \text{ and } \circ \text{ is given in Definition 5.}$$

Definition18. Let $\dot{\mathscr{G}} = (\mathcal{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_k)$ be an IFGS with the underlying GS $\mathscr{G}$. Then, $\dot{\rho}_i$-degree of a vertex $\mathbf{x} \in \mathbb{V}$ is expressed by $\mathbb{D}_{\dot{\rho}_i}(\mathbf{x}) = \langle \varkappa_{\mathbb{D}_{\dot{\rho}_i}}^m(\mathbf{x}), \varkappa_{\mathbb{D}_{\dot{\rho}_i}}^n(\mathbf{x}) \rangle$ [24],
where,

$$\varkappa_{\mathbb{D}_{\dot{\rho}_i}}^m(\mathbf{x}) = \sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{E}_i} \varkappa_{\dot{\rho}_i}^m(\mathbf{x}, \mathbf{y}),\ \text{and}\ \varkappa_{\mathbb{D}_{\dot{\rho}_i}}^n(\mathbf{x}) = \sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{E}_i} \varkappa_{\dot{\rho}_i}^n(\mathbf{x}, \mathbf{y}).$$

Definition19. Let $\dot{\mathscr{G}} = \big(\mathcal{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_k\big)$ be an IFGS with the underlying GS $\mathscr{G}$. The degree of the vertex $\mathbf{x} \in \mathbb{V}$ is symbolized by $\mathbb{D}(\mathbf{x}) = \big\langle \varkappa^m_{\mathbb{D}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x})\big\rangle$ and is defined as [24]:

$$\varkappa^m_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^m_{\mathbb{D}_{\dot{\rho}_i}}(\mathbf{x}), \text{ and } \varkappa^n_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^n_{\mathbb{D}_{\dot{\rho}_i}}(\mathbf{x}).$$

Definition20. Let $\dot{\mathscr{G}} = \big(\mathcal{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_k\big)$ be an IFGS with the underlying GS $\mathcal{G}$. Then, $\dot{\mathscr{G}}$ is called $\langle \mathbf{a}, \mathbf{b}\rangle$-$\dot{\rho}_i$ regular, if $\mathbb{D}_{\dot{\rho}_i}(\mathbf{x}) = (\mathbf{a}, \mathbf{b})$ and $\langle \mathbf{a}, \mathbf{b}\rangle$-regular, if $\mathbb{D}(\mathbf{x}) = \langle \mathbf{a}, \mathbf{b}\rangle\ \forall\ \mathbf{x} \in \mathbb{V}$ [24].

3 Linear Diophantine Fuzzy Graph Structures (LDFGS)

In this part, we initiate the theory of linear Diophantine fuzzy graph structure (LDFGS) and several fundamental ideas in LDFGSs, like $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, the strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size of an LDFGS, and the order of an LDFGS.

Definition21. Let $\mathfrak{L} = \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\big\rangle\Big)$ be an LDFS over $\mathbb{V}$, $\mathscr{G}$ be a GS and $\breve{\rho}_i \in \mathfrak{LDFS}(\mathbb{E}_i)$, $i \in \{1, 2, \cdots, k\}$. Then, $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\big)$ is said to be an LDFGS of GS $\mathscr{G}$, if $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{V}$:

$$\left.\begin{aligned}
\varkappa^m_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\leq \varkappa^m_{\mathfrak{L}}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{y}),\\
\varkappa^n_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\geq \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{y}),\\
\alpha_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\leq \alpha_{\mathfrak{L}}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}}(\mathbf{y}),\\
\beta_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\geq \beta_{\mathfrak{L}}(\mathbf{x}) \vee \beta_{\mathfrak{L}}(\mathbf{y}).
\end{aligned}\right\} \tag{18}$$

Example1. Let $\mathbb{V} = \big\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\big\}$, $\mathbb{E}_1 = \big\{(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_1, \mathbf{x}_3), (\mathbf{x}_3, \mathbf{x}_4)\big\}$, and $\mathbb{E}_2 = \big\{(\mathbf{x}_1, \mathbf{x}_4), (\mathbf{x}_2, \mathbf{x}_3), (\mathbf{x}_2, \mathbf{x}_4)\big\}$. Then, $\mathscr{G} = \big(\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2\big)$ is the GS shown in Figure 1. Define an LDFS $\mathfrak{L} \in \mathfrak{LDFS}(\mathbb{V})$ exhibited in Table 1.

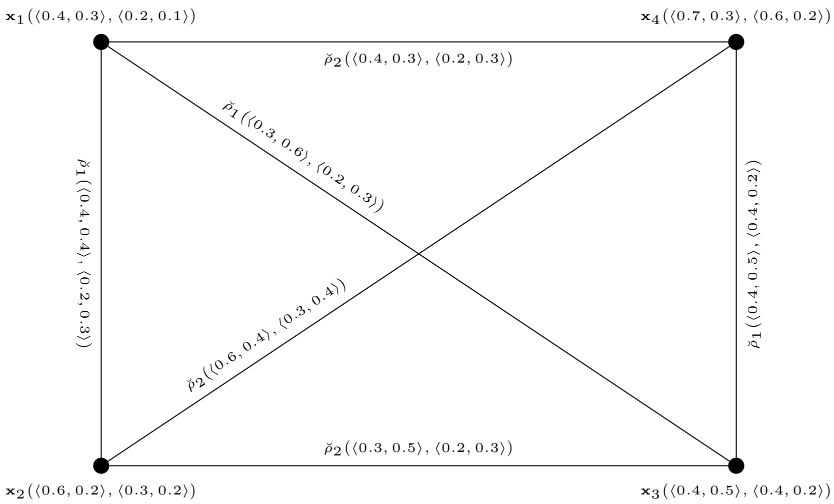

Figure 1. $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2\big)$

Table 1. Tabular representation of LDFS $\mathfrak{L}$

$\mathbb{V}$	$\Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

Let us take two LDFRs $\breve{\rho}_1, \breve{\rho}_2$ over $\mathbb{E}_1, \mathbb{E}_2$, respectively which are displayed in Table 2 and Table 3, respectively.

By routine calculations, it becomes evident that $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2\big)$ is an LDFGS of GS $\mathscr{G} = \big(\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2\big)$, which is depicted in Figure 1.

Table 2. $\breve{\rho}_1$

$\breve{\rho}_1$	$\Big(\big\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{x}_1,\mathbf{x}_2)$	$\big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_1,\mathbf{x}_3)$	$\big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_3,\mathbf{x}_4)$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$

Table 3. $\breve{\rho}_2$

$\breve{\rho}_2$	$\Big(\big\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{x}_1,\mathbf{x}_4)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_2,\mathbf{x}_3)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_2,\mathbf{x}_4)$	$\big(\langle 0.6, 0.4\rangle, \langle 0.3, 0.4\rangle\big)$

Definition22. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\big)$ be an LDFGS with underlying GS $\mathscr{G}$. When $(\mathbf{x},\mathbf{y}) \in \mathfrak{supp}(\breve{\rho}_i)$, then $(\mathbf{x},\mathbf{y})$ is termed $\breve{\rho}_i$-edge of $\breve{\mathscr{G}}$.

Example2. According to Example 1, $(\mathbf{x}_1,\mathbf{x}_4)$, $(\mathbf{x}_2,\mathbf{x}_3)$, $(\mathbf{x}_2,\mathbf{x}_4)$ are $\breve{\rho}_2$-edges as $\mathfrak{supp}(\breve{\rho}_2) = \big\{(\mathbf{x}_1,\mathbf{x}_4), (\mathbf{x}_2,\mathbf{x}_3), (\mathbf{x}_2,\mathbf{x}_4)\big\}$ and $(\mathbf{x}_1,\mathbf{x}_2)$, $(\mathbf{x}_1,\mathbf{x}_3)$, $(\mathbf{x}_3,\mathbf{x}_4)$ are $\breve{\rho}_1$-edges since $\mathfrak{supp}(\breve{\rho}_1) = \big\{(\mathbf{x}_1,\mathbf{x}_2), (\mathbf{x}_1,\mathbf{x}_3), (\mathbf{x}_3,\mathbf{x}_4)\big\}$.

Definition23. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ be an LDFGS with underlying GS $\mathscr{G}$. A $\breve{\rho}_i$-path of $\breve{\mathscr{G}}$ is a sequence of vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_l$ that are distinct except possibly $\mathbf{x}_0 = \mathbf{x}_l$, such that $(\mathbf{x}_{j-1}, \mathbf{x}_j)$ is a $\breve{\rho}_i$-edge $\forall\ j = 1, 2, 3, \cdots, l$.

Example3. In the light of Example 1, $\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2$ is a $\breve{\rho}_1$-path and $\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$ is a $\breve{\rho}_1$-path. Analogously, $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is a $\breve{\rho}_2$-path and $\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3$ is a $\breve{\rho}_2$-path.

Definition24. In an LDFGS $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ with underlying GS $\mathscr{G}$, two vertices $\mathbf{x}, \mathbf{y}$ of $\breve{\mathscr{G}}$ are said to be $\breve{\rho}_i$-connected, if they are joined by a $\breve{\rho}_i$-path.

Example4. From Example 1, all vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$- and $\breve{\rho}_2$-connected according to the Example 3 since they are joined by both $\breve{\rho}_1$- and $\breve{\rho}_2$- paths. Since $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{V}$ they are connected by $\breve{\rho}_i\ \forall\ i = 1, 2$, so $\breve{\mathscr{G}}$ is connected LDFGS because $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_3) > 0$, $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_2) > 0$, and $\breve{\rho}_1(\mathbf{x}_3, \mathbf{x}_4) > 0$ so, $\mathbf{x}_1$, $\mathbf{x}_3$ are $\breve{\rho}_1$-connected, $\mathbf{x}_1$, $\mathbf{x}_2$ are $\breve{\rho}_1$-connected, and $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$-connected, respectively. Similarly, $\mathbf{x}_2$, $\mathbf{x}_3$ are $\breve{\rho}_2$-connected, $\mathbf{x}_2$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected, and $\mathbf{x}_1$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected.

Definition25. Presume that $\mathfrak{P} = \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdot, \mathbf{x}_l$ be a $\breve{\rho}_i$-path of an LDFGS $\hat{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\big)$ with underlying GS $\mathscr{G}$. Then, the strength of the $\breve{\rho}_i$-path $\mathfrak{P}$, is postulated as:

$$\mathfrak{st}(\mathfrak{P}) = \Big(\big\langle \varkappa^m_{St(\mathfrak{P})}, \varkappa^n_{St(\mathfrak{P})}\big\rangle, \big\langle \alpha_{St(\mathfrak{P})}, \beta_{St(\mathfrak{P})}\big\rangle\Big), \tag{19}$$

where,

$$\left.\begin{aligned} \varkappa^m_{\mathfrak{st}(\mathfrak{P})} = \bigwedge_{j=1}^{k} \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \varkappa^n_{\mathfrak{st}(\mathfrak{P})} = \bigvee_{j=1}^{k} \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \\ \alpha_{\mathfrak{st}(\mathfrak{P})} = \bigwedge_{j=1}^{k} \alpha_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \beta_{\mathfrak{st}(\mathfrak{P})} = \bigvee_{j=1}^{k} \beta_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \end{aligned}\right\} \tag{20}$$

for $i = 1, 2, ..., k$.

Example5. (Continued from Example 3) We have observed that $\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2$ is a $\breve{\rho}_1$-path and $\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$ is a $\breve{\rho}_1$-path. Also, $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is a $\breve{\rho}_2$-path and $\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3$ is a $\breve{\rho}_2$-path. We can evaluate their strengths as follows:

Strength of $\breve{\rho}_1$-path $\mathfrak{P}_1 = \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2$:

$$\begin{aligned} \varkappa^m_{St(\mathfrak{P}_1)} &= \wedge^3_{j=2} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \wedge 0.3 = 0.3 \\ \varkappa^n_{St(\mathfrak{P}_1)} &= \vee^3_{j=2} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \vee 0.6 = 0.6 \\ \alpha_{St(\mathfrak{P}_1)} &= \wedge^3_{j=2} \alpha_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.2 \wedge 0.2 = 0.2 \\ \beta_{St(\mathfrak{P}_1)} &= \vee^3_{j=2} \beta_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.3 \vee 0.3 = 0.3 \end{aligned}$$

Thus, $\mathfrak{st}(\mathfrak{P}_1) = \big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$.

Likewise, we can determine strength of $\breve{\rho}_1$-path $\mathfrak{P}_2 = \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$ which is given by $\mathfrak{st}(\mathfrak{P}_2) = \big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$, strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is $\mathfrak{st}(\mathfrak{P}_3) = \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle\big)$ and strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3$ is an $\mathfrak{st}(\mathfrak{P}_4) = \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle\big)$.

Definition26. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS of GS $\mathscr{G}$. Then, $\breve{\rho}_i$-strength of connectedness of any two vertices $\mathbf{x}_1, \mathbf{x}_2$ is portrayed as:

$$\big(\breve{\rho}_i\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \Big(\Big\langle \big(\varkappa^m_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\varkappa^n_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle, \Big\langle \big(\alpha_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\beta_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle\Big), \tag{21}$$

where,

$$\left.\begin{aligned} \big(\varkappa^m_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigvee_{j=1}^{\infty} \big(\varkappa^m_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2), \text{ and } \big(\varkappa^n_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \bigwedge_{j=1}^{\infty} \big(\varkappa^n_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2).\\ \big(\alpha_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigvee_{j=1}^{\infty} \big(\alpha_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2), \text{ and } \big(\beta_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \bigwedge_{j=1}^{\infty} \big(\beta_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2).\end{aligned}\right\} \tag{22}$$

Here, $\big(\breve{\rho}_i\big)^j(\mathbf{x}_1, \mathbf{x}_2) = \Big(\Big\langle \big(\varkappa^m_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2), \big(\varkappa^n_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle, \Big\langle \big(\alpha_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2), \big(\beta_{\breve{\rho}_i}\big)^{j}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle\Big) = \big((\breve{\rho}_i)^{j-1} \circ \breve{\rho}_i\big)(\mathbf{x}_1, \mathbf{x}_2)$, and the composition $\circ$ among any two LDFRs is provided in Definition 8.

Example6. In Example 1, we can evaluate the terms as defined in above definition as follows:

$$\begin{aligned} (\varkappa^m_{\breve{\rho}_2})^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigvee_z \Big\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{z}) \wedge \varkappa^m_{\breve{\rho}_2}(\mathbf{z}, \mathbf{x}_2)\Big\} = \vee\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{x}_4) \wedge \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_4, \mathbf{x}_2)\} = 0.4 \wedge 0.6 = 0.4\\ (\varkappa^n_{\breve{\rho}_2})^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigwedge_z \{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{z}) \vee \varkappa^n_{\breve{\rho}_2}(\mathbf{z}, \mathbf{x}_2)\} = \wedge\{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{x}_4) \vee \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_4, \mathbf{x}_2)\} = 0.3 \vee 0.4 = 0.4\\ (\alpha_{\breve{\rho}_2})^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigvee_z \{\alpha_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{z}) \wedge \alpha_{\breve{\rho}_2}(\mathbf{z}, \mathbf{x}_2)\} = \vee\{\alpha_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{x}_4) \wedge \alpha_{\breve{\rho}_2}(\mathbf{x}_4, \mathbf{x}_2)\} = 0.2 \wedge 0.3 = 0.2\\ (\beta_{\breve{\rho}_2})^{\infty}(\mathbf{x}_1, \mathbf{x}_2) &= \bigwedge_z \{\beta_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{z}) \vee \beta_{\breve{\rho}_2}(\mathbf{z}, \mathbf{x}_2)\} = \wedge\{\beta_{\breve{\rho}_2}(\mathbf{x}_1, \mathbf{x}_4) \vee \beta_{\breve{\rho}_2}(\mathbf{x}_4, \mathbf{x}_2)\} = 0.3 \vee 0.4 = 0.4\end{aligned}$$

Therefore, $\big(\breve{\rho}_2\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.4\rangle\big)$.
Similarly, we one can find $\big(\breve{\rho}_2\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_3) = \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle\big)$ and $\big(\breve{\rho}_1\big)^{\infty}(\mathbf{x}_2, \mathbf{x}_3) = \big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$.

Definition27. Let $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. Then $\breve{\rho}_i$-degree of a vertex $\mathbf{x} \in \mathbb{V}$ is articulated as:

$$\mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle\Big), \tag{23}$$

where,

$$\left.\begin{aligned} \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) &= \sum_{i=1, \mathbf{x}\neq\mathbf{y}, (\mathbf{x},\mathbf{y})\in\mathbb{E}_i}^{k} \varkappa^m_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1, \mathbf{x}\neq\mathbf{y}, (\mathbf{x},\mathbf{y})\in\mathbb{E}_i}^{k} \varkappa^n_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}),\\ \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) &= \sum_{i=1, \mathbf{x}\neq\mathbf{y}, (\mathbf{x},\mathbf{y})\in\mathbb{E}_i}^{k} \alpha_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}), \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1, \mathbf{x}\neq\mathbf{y}, (\mathbf{x},\mathbf{y})\in\mathbb{E}_i}^{k} \beta_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}).\end{aligned}\right\} \tag{24}$$

Definition28. Let $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. Then the degree of the vertex $\mathbf{x} \in \mathbb{V}$ is characterized as:

$$\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{D}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x})\big\rangle\Big), \tag{25}$$

where,

$$\varkappa^m_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \alpha_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k} \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}). \tag{26}$$

Example7. If we recall Example 1, then from Definition 27, $\breve{\rho}_i$-degrees of vertices can be determine as follows:

$$\varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathbb{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.3 + 0.4 = 0.7$$

$$\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathbb{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.6 + 0.4 = 1$$

$$\alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathbb{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.2 + 0.2 = 0.4$$

$$\beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1 \neq \mathbf{y}, (\mathbf{x}_1, \mathbf{y}) \in \mathbb{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{y}) = \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) + \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) = 0.3 + 0.3 = 0.6$$

Therefore, $\mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$.
Similarly, we can evaluate $\breve{\rho}_1$- and $\breve{\rho}_2$-degrees of all $\mathbf{x} \in \mathbb{V}$ which are displayed in Table 4 and Table 5, respectively.

Table 4. $\mathbb{D}_{\breve{\rho}_1}$

$\mathbb{V}$	$\big(\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\rangle, \langle \alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\rangle\big)$
$\mathbf{x}_1$	$\big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big)$

Table 5. $\mathbb{D}_{\breve{\rho}_2}$

$\mathbb{V}$	$\big(\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\rangle, \langle \alpha_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\rangle\big)$
$\mathbf{x}_1$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big)$

Now, in the light of Definition 28, we calculate the degrees $\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{D}_{\breve{\rho}_i}(\mathbf{x})$ as follows:

$$\mathbb{D}(\mathbf{x}_1) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_2) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_3) = \mathbb{D}_{\breve{\rho}_3}(\mathbf{x}_1) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_3) = \big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big) + \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big) = \big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_4) = \mathbb{D}_{\breve{\rho}_4}(\mathbf{x}_1) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_4) = \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$$

which can be also be seen in Table 6.

Table 6. $\mathbb{D}(\mathbf{x})$

$\mathbb{V}$	$\big(\langle \varkappa^m_{\mathbb{D}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x})\rangle, \langle \alpha_{\mathbb{D}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x})\rangle\big)$
$\mathbf{x}_1$	$\big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$

Definition29. Suppose that $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. Then total $\breve{\rho}_i$-degree of a vertex $\mathbf{x} \in \mathbb{V}$ is defined as:

$$\mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) + \mathfrak{L}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle\Big), \tag{27}$$

where,

$$\left.\begin{aligned} \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^n_{\breve{\rho}_i}(\mathbf{x}), \\ \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \alpha_{\breve{\rho}_i}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \beta_{\breve{\rho}_i}(\mathbf{x}). \end{aligned}\right\} \tag{28}$$

Definition30. Let $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. Then, the total degree of the vertex $\mathbf{x} \in \mathbb{V}$ is postulated as:

$$\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \Big(\big\langle \varkappa_{\mathbb{TD}}^{m}(\mathbf{x}), \varkappa_{\mathbb{TD}}^{n}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\big\rangle\Big), \tag{29}$$

where,

$$\varkappa_{\mathbb{TD}}^{m}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa_{\mathbb{TD}_{\breve{\rho}_i}}^{m}(\mathbf{x}), \varkappa_{\mathbb{TD}}^{n}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa_{\mathbb{TD}_{\breve{\rho}_i}}^{n}(\mathbf{x}), \alpha_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}). \tag{30}$$

Example8. (Continued from Examples 1 and 7) We can evaluate the $\breve{\rho}_i$-degrees for each vertex $\mathbf{x} \in \mathbb{V}$ by using Definition 29 as follows:

$$\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathfrak{L}(\mathbf{x}_1) = (\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle) + (\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle) = (\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle),$$
$$\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathfrak{L}(\mathbf{x}_2) = (\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle) + (\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle) = (\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle),$$
$$\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathfrak{L}(\mathbf{x}_3) = (\langle 0.7, 1.1\rangle, \langle 0.6, 0.5\rangle) + (\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle) = (\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle),$$
$$\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathfrak{L}(\mathbf{x}_4) = (\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle) + (\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle) = (\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle),$$

which is also demonstrated in Table 7.

Table 7. $\mathbb{TD}_{\breve{\rho}_1}$

$\mathbb{V}$	$\Big(\big\langle \varkappa_{\mathbb{TD}_{\breve{\rho}_1}}^{m}(\mathbf{x}), \varkappa_{\mathbb{TD}_{\breve{\rho}_1}}^{n}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle)$
$\mathbf{x}_2$	$(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle)$
$\mathbf{x}_3$	$(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle)$
$\mathbf{x}_4$	$(\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle)$

Moreover, $\breve{\rho}_i$-degrees for each vertex $\mathbf{x} \in \mathbb{V}$ are calculated in Table 8.

Table 8. $\mathbb{TD}_{\breve{\rho}_2}$

$\mathbb{V}$	$\Big(\big\langle \varkappa_{\mathbb{TD}_{\breve{\rho}_2}}^{m}(\mathbf{x}), \varkappa_{\mathbb{TD}_{\breve{\rho}_2}}^{n}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$(\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle)$
$\mathbf{x}_2$	$(\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle)$
$\mathbf{x}_3$	$(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle)$
$\mathbf{x}_4$	$(\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle)$

Now, from Definition 30, $\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x})$ are calculated as follows:

$$\mathbb{TD}(\mathbf{x}_1) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_1) = (\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle) + (\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle) = (\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle),$$
$$\mathbb{TD}(\mathbf{x}_2) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_2) = (\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle) + (\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle) = (\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle),$$
$$\mathbb{TD}(\mathbf{x}_3) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_3) = (\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle) + (\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle) = (\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle),$$
$$\mathbb{TD}(\mathbf{x}_4) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_4) = (\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle) + (\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle) = (\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle),$$

which is also showcased in Table 9.

Definition31. Let $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. Then order of $\breve{\mathscr{G}}$ is described as follows:

$$\mathcal{O}(\breve{\mathscr{G}}) = \Bigg(\Big\langle \sum_{\mathbf{x}\in\mathbb{V}} \varkappa_{\mathfrak{L}}^{m}(\mathbf{x}), \sum_{\mathbf{x}\in\mathbb{V}} \varkappa_{\mathfrak{L}}^{n}(\mathbf{x})\Big\rangle, \Big\langle \sum_{\mathbf{x}\in\mathbb{V}} \alpha_{\mathfrak{L}}(\mathbf{x}), \sum_{\mathbf{x}\in\mathbb{V}} \beta_{\mathfrak{L}}(\mathbf{x})\Big\rangle\Bigg). \tag{31}$$

Table 9. $\mathbb{TD}$

$\mathbb{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{TD}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle\big)$

Example9. If we revisit the Example 1, then we can calculate $\mathcal{O}(\breve{\mathscr{G}})$ as follows:

$$\begin{aligned}
\sum_{\mathbf{x}\in\mathbb{V}} \varkappa^m_{\mathfrak{L}}(\mathbf{x}) &= 0.4 + 0.6 + 0.4 + 0.7 = 2,\\
\sum_{\mathbf{x}\in\mathbb{V}} \varkappa^n_{\mathfrak{L}}(\mathbf{x}) &= 0.3 + 0.2 + 0.5 + 0.3 = 1.3,\\
\sum_{\mathbf{x}\in\mathbb{V}} \alpha_{\mathfrak{L}}(\mathbf{x}) &= 0.2 + 0.3 + 0.4 + 0.6 = 1.5,\\
\sum_{\mathbf{x}\in\mathbb{V}} \beta_{\mathfrak{L}}(\mathbf{x}) &= 0.1 + 0.2 + 0.2 + 0.2 = 0.7.
\end{aligned}$$

Therefore, $\mathcal{O}(\breve{\mathscr{G}}) = \big(\langle 2, 1.3\rangle, \langle 1.5, 0.7\rangle\big)$.

Definition32. Let $\breve{\mathscr{G}}$ be an LDFGS with underlying GS $\mathscr{G}$. The $\breve{\rho}_i$-size of $\breve{\mathscr{G}}$ is postulated as:

$$\mathfrak{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}) = \Big(\big\langle \varkappa^m_{\mathfrak{S}_{\breve{\rho}_i}}, \varkappa^n_{\mathfrak{S}_{\breve{\rho}_i}}\big\rangle, \big\langle \alpha_{\mathfrak{S}_{\breve{\rho}_i}}, \beta_{\mathfrak{S}_{\breve{\rho}_i}}\big\rangle\Big), \tag{32}$$

where,

$$\begin{aligned}
\varkappa^m_{\mathfrak{S}_{\breve{\rho}_i}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_i} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\ \varkappa^n_{\mathfrak{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_i} \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\\
\alpha_{\mathfrak{S}_{\breve{\rho}_i}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_i} \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\ \beta_{\mathfrak{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_i} \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}).
\end{aligned} \tag{33}$$

Additionally, the size of $\breve{\mathscr{G}}$ is denoted and characterized as:

$$\mathfrak{S}(\breve{\mathscr{G}}) = \sum_{i=1}^{n} \mathfrak{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}). \tag{34}$$

Example10. If we revisit Example 1, then

$$\begin{aligned}
\varkappa^m_{\mathfrak{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.3 + 0.4 = 1.1,\\
\varkappa^n_{\mathfrak{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.6 + 0.5 = 1.5,\\
\alpha_{\mathfrak{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.2 + 0.2 + 0.4 = 0.8,\\
\beta_{\mathfrak{S}_{\breve{\rho}_1}} &= \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.3 + 0.3 + 0.2 = 0.8.
\end{aligned}$$

Therefore, $\mathfrak{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) = \big(\langle \varkappa^m_{\mathfrak{S}_{\breve{\rho}_1}}, \varkappa^n_{\mathfrak{S}_{\breve{\rho}_1}}\rangle, \langle \alpha_{\mathfrak{S}_{\breve{\rho}_1}}, \beta_{\mathfrak{S}_{\breve{\rho}_1}}\rangle\big) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big)$. Similarly, $\mathfrak{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big)$.

Moreover, the size of $\breve{\mathscr{G}}$ is evaluated as:

$$\mathfrak{S}(\breve{\mathscr{G}}) = \mathfrak{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) + \mathfrak{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big) + \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big) = \big(\langle 2.4, 2.7\rangle, \langle 1.8, 1.5\rangle\big).$$

4 Maximal Product of Two LDFGSs

This segment the idea of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of a vertex in maximal product. Also, certain results related to these concepts are proved with several concrete illustrations.

Definition33. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_n\big)$ be two LDFGSs of the GSs $\mathscr{G}_1 = \big(\mathbb{V}_1, \mathbb{E}'_1, \mathbb{E}'_2, \cdots, \mathbb{E}'_n\big)$ and $\mathscr{G}_2 = \big(\mathbb{V}_2, \mathbb{E}''_1, \mathbb{E}''_2, \cdots, \mathbb{E}''_n\big)$, respectively. Then, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ is called maximal LDFGS with underlying crisp GS $\mathscr{G} = \big(\mathbb{V}, \mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_n\big)$, where $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ and $\mathbb{E}_i = \big\{\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) : \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i \text{ or } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i\big\}$. The LDF vertex set $\mathfrak{L}$ and LDFRs $\breve{\rho}_i$ in maximal product $\breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ are defined as:

$$\begin{aligned}
\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2 &= \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big) * \Big(\big\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}_2}(\mathbf{y}), \beta_{\mathfrak{L}_2}(\mathbf{y})\big\rangle\Big) \\
&= \Big(\big\langle (\varkappa^m_{\mathfrak{L}_1} * \varkappa^m_{\mathfrak{L}_2})(\mathbf{x}, \mathbf{y}), (\varkappa^n_{\mathfrak{L}_1} * \varkappa^n_{\mathfrak{L}_2})(\mathbf{x}, \mathbf{y})\big\rangle, \big\langle (\alpha_{\mathfrak{L}_1} * \alpha_{\mathfrak{L}_2})(\mathbf{x}, \mathbf{y}), (\beta_{\mathfrak{L}_1} * \beta_{\mathfrak{L}_2})(\mathbf{x}, \mathbf{y})\big\rangle\Big) \\
&= \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\big\rangle\Big), \qquad (35)
\end{aligned}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}), \\
\varkappa^n_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}) &= \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}), \\
\alpha_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}) &= \alpha_{\mathfrak{L}_1}(\mathbf{x}) \vee \alpha_{\mathfrak{L}_2}(\mathbf{y}), \\
\beta_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}) &= \beta_{\mathfrak{L}_1}(\mathbf{x}) \wedge \beta_{\mathfrak{L}_2}(\mathbf{y}),
\end{aligned}\right\} \qquad (36)$$

$\forall\, (\mathbf{x}, \mathbf{y}) \in \mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ and $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ are defined as :

$$\begin{aligned}
\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i &= \Big(\big\langle \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{y}_1), \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{y}_1)\big\rangle, \big\langle \alpha_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{y}_1), \beta_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{y}_1)\big\rangle\Big) \\
&\quad * \Big(\big\langle \varkappa^m_{\breve{\rho}''_i}(\mathbf{x}_2, \mathbf{y}_2), \varkappa^n_{\breve{\rho}''_i}(\mathbf{x}_2, \mathbf{y}_2)\big\rangle, \big\langle \alpha._{\breve{\rho}''_i}(\mathbf{x}_2, \mathbf{y}_2), \beta_{\breve{\rho}''_i}(\mathbf{x}_2, \mathbf{y}_2)\big\rangle\Big) \\
&= \Big(\big\langle (\varkappa^m_{\breve{\rho}'_i} * \varkappa^m_{\breve{\rho}''_i})(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), (\varkappa^n_{\breve{\rho}'_i} * \varkappa^n_{\breve{\rho}''_i})(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\big\rangle, \\
&\qquad \big\langle (\alpha_{\breve{\rho}'_i} * \alpha_{\breve{\rho}''_i})(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), (\beta_{\breve{\rho}'_i} * \beta_{\breve{\rho}''_i})(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\big\rangle\Big) \\
&= \Big(\big\langle \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\big\rangle\Big), \qquad (37)
\end{aligned}$$

where,

$$\varkappa^m_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2) = \begin{cases} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\breve{\rho}''_i}(\mathbf{y}_1, \mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i \\ \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i \end{cases} \qquad (38)$$

$$\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \begin{cases} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_1, \mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i \\ \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i \end{cases} \qquad (39)$$

$$\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \begin{cases} \alpha_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \alpha_{\breve{\rho}''_i}(\mathbf{y}_1, \mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i \\ \alpha_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \alpha_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i \end{cases} \qquad (40)$$

$$\beta_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \begin{cases} \beta_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \beta_{\breve{\rho}''_i}(\mathbf{y}_1, \mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i \\ \beta_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \beta_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i \end{cases} \qquad (41)$$

$$i = 1, 2, \cdots, n.$$

Example11. Let us consider two LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$, which is showcased in Figure 2 with underlying GSs $\mathscr{G}_1 = (\mathbb{V}_1, \mathbb{E}'_1, \mathbb{E}'_2, \mathbb{E}'_3)$ and $\mathscr{G}_2 = (\mathbb{V}_2, \mathbb{E}''_1)$, respectively, where $\mathbb{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathbb{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ are two sets of vertices and $\mathbb{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_3)\}$, $\mathbb{E}'_2 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$, and $\mathbb{E}'_3 = \{(\mathbf{u}_2, \mathbf{u}_3)\}$ are the set of edges on $\mathbb{V}_1$, and $\mathbb{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$ is the edges set on $\mathbb{V}_2$ such that $\mathbb{E}'_i$ and $\mathbb{E}''_i$ are irreflexive and symmetric binary relations on $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively. The LDFSs $\mathfrak{L}_1$ on $\mathbb{V}_1$ and $\mathfrak{L}_2$ on $\mathbb{V}_2$ are given in the Table 10 and Table 11, respectively. The LDFRs $\breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3$ over the $\mathbb{E}'_1, \mathbb{E}'_2, \mathbb{E}'_3$, and $\breve{\rho}''_1$ over $\mathbb{E}''_1$ given in Tables 12-15, respectively. By using Definition 33, we obtain the following LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ illustrated in Figure 3 and shown in Table 16 and LDFRs $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ for $i = 1, 2, 3$ shown in Tables 17-19, respectively.

Table 10. LDFS $\mathfrak{L}_1$

$\mathbb{V}_1$	$\big(\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$\mathbf{u}_3$	$\big(\langle 0.8, 0.9\rangle, \langle 0.6, 0.3\rangle\big)$

Table 11. LDFS $\mathfrak{L}_2$

$\mathbb{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.7, 0.4\rangle, \langle 0.3, 0.2\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.3, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 12. $\breve{\rho}'_1$

$\mathbb{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_3)$	$\big(\langle 0.6, 0.9\rangle, \langle 0.4, 0.5\rangle\big)$

Table 13. $\breve{\rho}'_2$

$\mathbb{E}'_2$	$\big(\langle \varkappa^m_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.4, 0.5\rangle, \langle 0.3, 0.4\rangle\big)$

Table 14. $\breve{\rho}'_3$

$\mathbb{E}'_3$	$\big(\langle \varkappa^m_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle\rangle\big)$
$(\mathbf{u}_2, \mathbf{u}_3)$	$\big(\langle 0.4, 0.9\rangle, \langle 0.5, 0.4\rangle\big)$

Table 15. $\breve{\rho}''_1$

$\mathbb{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$

Table 16. $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathbb{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.7, 0.3\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3, \mathbf{v}_1)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_3, \mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.6, 0.1\rangle\big)$

Table 17. $\breve{\rho}_1$

$\mathbb{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.3, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$(\mathbf{u}_3\mathbf{v}_1, \mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Definition34. An LDFGS $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ is said to be $\breve{\rho}_i$-strong, if

$$\left.\begin{aligned} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \varkappa^m_{\mathfrak{L}}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{y}),\\ \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{y}),\\ \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \alpha_{\mathfrak{L}}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}}(\mathbf{y}),\\ \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}) &= \beta_{\mathfrak{L}}(\mathbf{x}) \vee \beta_{\mathfrak{L}}(\mathbf{y}), \end{aligned}\right\} \tag{42}$$

Table 18. $\breve{\rho}_2$

$\mathbb{E}_2$	$\big(\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_2, \mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 19. $\breve{\rho}_3$

$\mathbb{E}_3$	$\big(\langle \varkappa^m_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.2\rangle\big)$

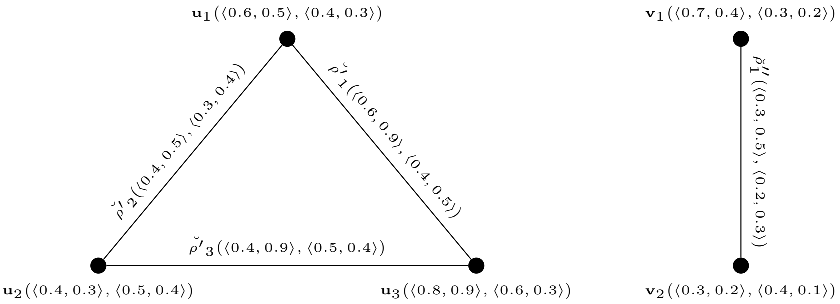

Figure 2. LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$

$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{V}$. When $\breve{\mathscr{G}}$ is $\breve{\rho}_i$-strong $\forall\, i = 1, 2, \cdots, n$, then $\breve{\mathscr{G}}$ is a named strong LDFGS.

Proposition1. The maximal product of two strong LDFGSs is also a strong LDFGS.

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_n\big)$ be two strong LDFGSs. Then, in the light of Definition 33, we have the following cases:

Case 1: If $\mathbf{x}_1 = \mathbf{x}_2$ and $(\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{E}''_i$. Then,

$$\begin{aligned}
\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\rho''_i}(\mathbf{y}_1, \mathbf{y}_2) \\
&= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\big] \\
&= \big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\big] \wedge \big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\big] \\
&= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2).
\end{aligned}$$

Analogously, we can show that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \wedge \alpha_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \vee \beta_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$.

Case 2: If $\mathbf{y}_1 = \mathbf{y}_2$ and $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{E}'_i$, we have

$$\begin{aligned}
\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1, \mathbf{x}_2) \\
&= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big] \\
&= \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\big] \wedge \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big] \\
&= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2).
\end{aligned}$$

In the same way, we can prove that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \wedge \alpha_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \vee \beta_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2)$. Thus, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is a strong LDFGS.

Proposition2. The maximal product of two connected LDFGSs is a connected LDFGS.

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_n\big)$ be two connected LDFGSs with underlying GSs $\mathscr{G}_1 = \big(\mathbb{V}_1, \mathbb{E}'_1, \mathbb{E}'_2, \cdots, \mathbb{E}'_n\big)$ and $\mathscr{G}_2 = \big(\mathbb{V}_2, \mathbb{E}''_1, \mathbb{E}''_2, \cdots, \mathbb{E}''_n\big)$, respectively. Assume that

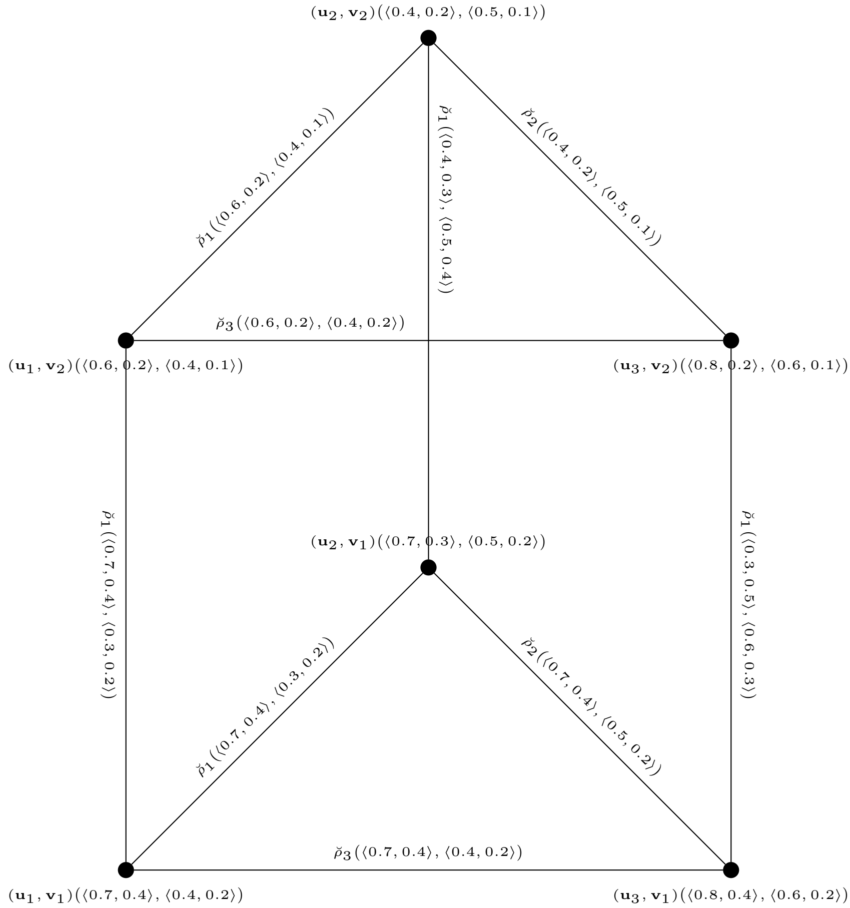

Figure 3. Maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

$\mathbb{V}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m\}$ and $\mathbb{V}_2 = \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$. Then,

$$(\varkappa^m_{\breve{\rho'}_i})^\infty(\mathbf{x}_i, \mathbf{x}_j) > 0 \text{ and } (\varkappa^m_{\breve{\rho''}_i})^\infty(\mathbf{y}_i, \mathbf{y}_j) > 0;$$
$$(\varkappa^n_{\breve{\rho'}_i})^\infty(\mathbf{x}_i, \mathbf{x}_j) > 0 \text{ and } (\varkappa^n_{\breve{\rho''}_i})^\infty(\mathbf{y}_i, \mathbf{y}_j) > 0;$$
$$(\alpha_{\breve{\rho'}_i})^\infty(\mathbf{x}_i, \mathbf{x}_j) > 0 \text{ and } (\alpha_{\breve{\rho''}_i})^\infty(\mathbf{y}_i, \mathbf{y}_j) > 0;$$
$$(\beta_{\breve{\rho'}_i})^\infty(\mathbf{x}_i, \mathbf{x}_j) > 0 \text{ and } (\beta_{\breve{\rho''}_i})^\infty(\mathbf{y}_i, \mathbf{y}_j) > 0,$$

$\forall\, \mathbf{x}_i, \mathbf{x}_j \in \mathbb{V}_1$ and $\mathbf{y}_i, \mathbf{y}_j \in \mathbb{V}_2$. Consider m subgraphs of $\mathscr{G}$ with the vertex sets $\big\{(\mathbf{x}_i, \mathbf{y}_1), (\mathbf{x}_i, \mathbf{y}_2), \cdots, (\mathbf{x}_i, \mathbf{y}_n)\big\}$ for $i = 1, 2, \cdots, m$. Each of these subgraphs of $\mathscr{G}$ is connected since $\mathbf{x}_i$'s are the same and $\mathscr{G}_2$ is connected, each $\mathbf{y}_i$ is adjacent to at least one of the vertices in $\mathbb{V}_2$. Since $\mathscr{G}_1$ is connected, each $\mathbf{x}_i$ is also adjacent to at least one of the vertices in $\mathbb{V}_1$. So, there exists one edge between any pair of the above m subgraphs. Therefore, it follows that

$$(\varkappa^m_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i, \mathbf{y}_j), (\mathbf{x}_k, \mathbf{y}_l)\big) > 0, (\varkappa^n_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i, \mathbf{y}_j), (\mathbf{x}_k, \mathbf{y}_l)\big) > 0, \text{ and}$$
$$(\alpha_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i, \mathbf{y}_j), (\mathbf{x}_k, \mathbf{y}_l)\big) > 0, (\beta_{\breve{\rho'}_i})^\infty\big((\mathbf{x}_i, \mathbf{y}_j), (\mathbf{x}_k, \mathbf{y}_l)\big) > 0,$$

$\forall\, ((\mathbf{x}_i, \mathbf{y}_j), (\mathbf{x}_k, \mathbf{y}_l)) \in \mathbb{E}_i$. Hence, $\breve{\mathscr{G}}$ is connected LDFGS.

Definition35. Suppose that $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ be the maximal LDFGS of LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_n\big)$. Then, the degree of a vertex in $\breve{\mathscr{G}}$ is described as:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{43}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \beta_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\}\quad (44)$$

Also, $\breve{\rho}_i - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$ of a vertex $(\mathbf{x}_i,\mathbf{y}_j)$ of maximal product $\mathscr{G}$ is articulated as follows:

$$\breve{\rho}_i - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_i - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_i - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_i - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_i - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \quad (45)$$

where,

$$\left.\begin{aligned}
\varkappa^m_i - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\rho''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_i - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_i - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_i - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \beta_{\mathbb{E}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\}\quad (46)$$

Example12. (Continued from Example 11) With the same LDFGSs $\breve{\mathscr{G}}_1$, $\breve{\mathscr{G}}_2$ and their maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ with underlying GSs $\mathscr{G}_1$, $\mathscr{G}_2$ and their maximal product $\mathscr{G} = \mathscr{G}_1 * \mathscr{G}_2$. From Definition 35, the degrees of vertices in $\breve{\mathscr{G}}$ are computed as follows:

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee 0.7 + 0.6\vee 0.7 + 0.3\vee 0.6 = 2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee 0.3 + 0.6\vee 0.3 + 0.3\vee 0.6 = 1.6\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee 0.7 + 0.4\vee 0.7 + 0.3\vee 0.4 = 1.8\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee 0.3 + 0.4\vee 0.3 + 0.3\vee 0.4 = 1.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee 0.7 + 0.6\vee 0.7 + 0.3\vee 0.8 = 2.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee 0.3 + 0.6\vee 0.3 + 0.3\vee 0.8 = 1.8
\end{aligned}$$

Analogously,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.4 + 0.9\wedge 0.4 + 0.5\wedge 0.5 = 1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.2 + 0.9\wedge 0.2 + 0.5\wedge 0.5 = 0.9
\end{aligned}$$

$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.5 \wedge 0.4 + 0.9 \wedge 0.4 + 0.5 \wedge 0.3 = 1.1$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.5 \wedge 0.2 + 0.9 \wedge 0.2 + 0.5 \wedge 0.3 = 0.7$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.9 \wedge 0.4 + 0.9 \wedge 0.4 + 0.5 \wedge 0.9 = 1.3$$
$$\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.9 \wedge 0.2 + 0.9 \wedge 0.2 + 0.5 \wedge 0.9 = 0.9$$

Also,

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_1)$$
$$= 0.3 \vee 0.3 + 0.4 \vee 0.3 + 0.2 \vee 0.4 = 1.1$$
$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_1)$$
$$= 0.3 \vee 0.4 + 0.4 \vee 0.4 + 0.2 \vee 0.4 = 1.2$$
$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.3 \vee 0.3 + 0.5 \vee 0.3 + 0.2 \vee 0.5 = 1.3$$
$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.3 \vee 0.4 + 0.5 \vee 0.4 + 0.2 \vee 0.5 = 1.4$$
$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \alpha_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.5 \vee 0.3 + 0.4 \vee 0.3 + 0.2 \vee 0.6 = 1.5$$
$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \alpha_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.5 \vee 0.4 + 0.4 \vee 0.4 + 0.2 \vee 0.6 = 1.5$$

Further,

$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \beta_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_1)$$
$$= 0.4 \wedge 0.2 + 0.5 \wedge 0.2 + 0.3 \wedge 0.3 = 0.7$$
$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \beta_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_1)$$
$$= 0.4 \wedge 0.1 + 0.5 \wedge 0.1 + 0.3 \wedge 0.3 = 0.5$$
$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \beta_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.4 \wedge 0.2 + 0.4 \wedge 0.2 + 0.3 \wedge 0.4 = 0.7$$
$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \beta_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_2)$$
$$= 0.4 \wedge 0.1 + 0.4 \wedge 0.1 + 0.3 \wedge 0.4 = 0.5$$
$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \beta_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.4 \wedge 0.2 + 0.5 \wedge 0.2 + 0.3 \wedge 0.3 = 0.7$$
$$\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \beta_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_3)$$
$$= 0.4 \wedge 0.1 + 0.5 \wedge 0.1 + 0.3 \wedge 0.3 = 0.5$$

These degrees are also exhibited in Table 20.
Now, we calculate $\breve{\rho}_i - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)\ \forall\ i = 1,2,3$ as follows:

$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.4 \vee 0.7 + 0.3 \vee 0.6 = 1.3$$
$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.4 \vee 0.3 + 0.3 \vee 0.6 = 1$$
$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.4 \vee 0.7 + 0.3 \vee 0.4 = 1.1$$
$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.4 \vee 0.3 + 0.3 \vee 0.4 = 0.8$$
$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.3 \vee 0.8 = 0.8$$
$$\varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.3 \vee 0.8 = 0.8$$

Table 20. $\mathbb{D}_{\breve{\mathscr{G}}}$

$\mathbb{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 2, 0.9\rangle, \langle 1.1, 0.7\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 1.6, 1.1\rangle, \langle 1.2, 0.5\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 1.8, 0.7\rangle, \langle 1.3, 0.7\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 1.2, 1.3\rangle, \langle 1.4, 0.5\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 2.2, 0.9\rangle, \langle 1.5, 0.7\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 1.3\rangle, \langle 1.5, 0.5\rangle\big)$

Similarly, $\varkappa^n_i - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ can be calculated as:

$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.4 + 0.5 \wedge 0.5 = 0.9$$
$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.2 + 0.5 \wedge 0.5 = 0.7$$
$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.4 + 0.5 \wedge 0.3 = 0.7$$
$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.2 + 0.5 \wedge 0.3 = 0.5$$
$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.5 = 0.5$$

Moreover, $\alpha_i - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ are evaluated as follows:

$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.3 \vee 0.3 + 0.2 \vee 0.4 = 0.7$$
$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.3 \vee 0.4 + 0.2 \vee 0.4 = 0.8$$
$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.3 \vee 0.3 + 0.2 \vee 0.5 = 0.8$$
$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \alpha_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) + \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.3 \vee 0.4 + 0.2 \vee 0.5 = 0.9$$
$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.2 \vee 0.6 = 0.6$$
$$\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \alpha_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \vee \alpha_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.2 \vee 0.6 = 0.6$$

Also, $\beta_i - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ are calculated as:

$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) = \beta_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.4 \wedge 0.2 + 0.3 \wedge 0.3 = 0.5$$
$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) = \beta_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.4 \wedge 0.1 + 0.3 \wedge 0.3 = 0.4$$
$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) = \beta_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.4 \wedge 0.2 + 0.3 \wedge 0.4 = 0.5$$
$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) = \beta_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) + \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.4 \wedge 0.1 + 0.3 \wedge 0.4 = 0.4$$
$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) = \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.3 \wedge 0.3 = 0.3$$
$$\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) = \beta_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2) \wedge \beta_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.3 \wedge 0.3 = 0.3$$

These degrees are also outlined in Table 21.

Table 21. $\breve{\rho}_1 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathbb{V}$	$\Big(\big\langle \varkappa^m_1 - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \varkappa^n_1 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_1 - \mathbf{d}_{\alpha_{\mathfrak{L}}}(\mathbf{x},\mathbf{y}), \beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 1.3, 0.9\rangle, \langle 0.7, 0.5\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 1, 0.7\rangle, \langle 0.8, 0.4\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 1.1, 0.7\rangle, \langle 0.8, 0.5\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.8, 0.5\rangle, \langle 0.9, 0.4\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$

Similarly, we can calculate $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}$ as:

$$\begin{aligned}
\varkappa_2^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_2, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_2'}^m(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7 \\
\varkappa_2^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_2, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_2'}^m(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4 \\
\varkappa_2^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_3, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_2'}^m(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7 \\
\varkappa_2^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_3, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_2'}^m(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4 \\
\varkappa_2^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_2, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_2'}^n(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4 \\
\varkappa_2^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_2, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_2'}^n(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2 \\
\varkappa_2^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_3, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_2'}^n(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4 \\
\varkappa_2^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_3, \mathbf{v}_2) &= \varkappa_{\rho_2'}^n(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2 \\
\alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) &= \alpha_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.5 \vee 0.3 = 0.5 \\
\alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) &= \alpha_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.5 \vee 0.4 = 0.5 \\
\alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) &= \alpha_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.5 \vee 0.3 = 0.5 \\
\alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) &= \alpha_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.5 \vee 0.4 = 0.5 \\
\beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) &= \beta_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \wedge 0.2 = 0.2 \\
\beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) &= \beta_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \wedge 0.1 = 0.1 \\
\beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) &= \beta_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \wedge 0.2 = 0.2 \\
\beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) &= \beta_{\breve{\rho}_2'}(\mathbf{u}_2, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \wedge 0.1 = 0.1
\end{aligned}$$

These degrees are also tabulated in Table 22.

Table 22. $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathbb{V}$	$\Big(\big\langle \varkappa_2^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{x},\mathbf{y}), \varkappa_2^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3, \mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_3, \mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Likewise, we can determine $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}$ as:

$$\begin{aligned}
\varkappa_3^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_1, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_3'}^m(\mathbf{u}_1, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_1) = 0.6 \vee 0.7 = 0.7 \\
\varkappa_3^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_1, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_3'}^m(\mathbf{u}_1, \mathbf{u}_3) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_2) = 0.6 \vee 0.3 = 0.6 \\
\varkappa_3^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_3, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_3'}^m(\mathbf{u}_3, \mathbf{u}_1) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_1) = 0.6 \vee 0.7 = 0.7 \\
\varkappa_3^m - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^m(\mathbf{u}_3, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_3'}^m(\mathbf{u}_3, \mathbf{u}_1) \vee \varkappa_{\mathfrak{L}_2}^m(\mathbf{v}_2) = 0.6 \vee 0.7 = 0.7 \\
\varkappa_3^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_1, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_3'}^n(\mathbf{u}_1, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4 \\
\varkappa_3^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_1, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_3'}^n(\mathbf{u}_1, \mathbf{u}_3) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2 \\
\varkappa_3^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_3, \mathbf{v}_1) &= \varkappa_{\breve{\rho}_3'}^n(\mathbf{u}_3, \mathbf{u}_1) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4 \\
\varkappa_3^n - \varkappa_{\mathbb{D}_{\breve{\mathscr{G}}}}^n(\mathbf{u}_3, \mathbf{v}_2) &= \varkappa_{\breve{\rho}_3'}^n(\mathbf{u}_3, \mathbf{u}_1) \wedge \varkappa_{\mathfrak{L}_2}^n(\mathbf{v}_2) = 0.4 \wedge 0.2 = 0.2 \\
\alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) &= \alpha_{\breve{\rho}_3'}(\mathbf{u}_1, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.3 = 0.4 \\
\alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) &= \alpha_{\breve{\rho}_3'}(\mathbf{u}_1, \mathbf{u}_3) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.4 = 0.4 \\
\alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) &= \alpha_{\breve{\rho}_3'}(\mathbf{u}_3, \mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.3 = 0.4 \\
\alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) &= \alpha_{\breve{\rho}_3'}(\mathbf{u}_3, \mathbf{u}_1) \vee \alpha_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.4 = 0.4 \\
\beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) &= \beta_{\breve{\rho}_3'}(\mathbf{u}_1, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.5 \wedge 0.2 = 0.2 \\
\beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) &= \beta_{\breve{\rho}_3'}(\mathbf{u}_1, \mathbf{u}_3) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.5 \wedge 0.1 = 0.1 \\
\beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) &= \beta_{\breve{\rho}_3'}(\mathbf{u}_3, \mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.5 \wedge 0.2 = 0.2 \\
\beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) &= \beta_{\breve{\rho}_3'}(\mathbf{u}_3, \mathbf{u}_1) \wedge \beta_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.5 \wedge 0.1 = 0.1
\end{aligned}$$

These degrees are summarized in Table 23.

Table 23. $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathbb{V}$	$\left(\langle \varkappa_3^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \varkappa_3^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y}), \beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x},\mathbf{y})\rangle\right)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$(\langle 0.7, 0.2\rangle, \langle 0.4, 0.1\rangle)$

Proposition3. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \left(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\right)$ is given as:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \left(\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle, \langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\right), \tag{47}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}\right\} \tag{48}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, then $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Therefore,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{G_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)
\end{aligned}$$

By adopting the same procedure, we can prove that

$$\mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{G_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{G_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

Proposition4. If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ are two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\left(\langle a, b\rangle, \langle c, d\rangle\right)$, where $a, b, c, d \in [0, 1]$ are fixed, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given as:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \left(\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle, \langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\right), \tag{49}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}\right\} \tag{50}$$

Proof. Let $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, then $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$ and $\mathfrak{L}_2$ is a constant LDFS. Thus,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i) a + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i) b + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}$$

Likewise, we can prove that

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i) c + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i) \beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

Proposition5. If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) \big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) \big\rangle \Big), \tag{51}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i), \\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i), \\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_1}(\mathbf{y}_j) \alpha_{\mathfrak{L}_1}(\mathbf{x}_i), \\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \beta_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}\right\} \tag{52}$$

Proof. Let $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, then $\breve{\rho}''_i \subseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$. So,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) \\
&= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Similarly, we one can that

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \alpha_{\mathfrak{L}_1}(\mathbf{x}_i) \text{ and } \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j) \beta_{\mathfrak{L}_1}(\mathbf{x}_i).$$

Proposition6. If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho'}_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $(\langle a, b\rangle, \langle c, d\rangle)$, where $a, b, c, d \in [0, 1]$ are fixed, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{53}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j)a,\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j)b,\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j)c,\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j)d.
\end{aligned}\right\} \tag{54}$$

Proof. Let $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho'}_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $(\langle a, b\rangle, \langle c, d\rangle)$. Therefore,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{G_2}(\mathbf{y}_j)a.
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b.
\end{aligned}$$

Similarly, it can be shown that

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c \ \text{ and } \ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d.$$

Proposition7. If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k)$ are two LDFGSs such that $\breve{\rho''}_i \subseteq \mathfrak{L}_1$ and $\breve{\rho'}_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is characterized as:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{55}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}\right\} \tag{56}$$

Proof. Let $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k)$ be two LDFGSs such that such that $\breve{\rho''}_i \subseteq \mathfrak{L}_1$ and $\breve{\rho'}_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Then,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{G_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}$$

Similarly, we can show that
$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i)$.

Proposition8. If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ are two LDFGSs, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is described as:

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{57}$$

where,

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{y}_j),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}\right\} \tag{58}$$

Proof. Let $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ be two LDFGSs such that such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, then $\breve{\rho}'_i \supseteq \mathfrak{L}_2$ and $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ $i = 1, 2, \cdots, k$. We have,

$$\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^m_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) + \big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\big]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \big(\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\big)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^n_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathbb{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathbb{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) + \big[\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\big]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \big(\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\big)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Similarly, we can show that

$\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$ and $\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$.

Proposition9. If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ are two LDFGSs, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $(\langle a, b\rangle, \langle c, d\rangle)$, where $a, b, c, d \in [0, 1]$ are fixed, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is characterized as:

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{59}$$

where,

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a,\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b,\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c,\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d.\end{aligned}\right\} \tag{60}$$

Proof. Analogous to the proof of Theorems 4 and 8.

Proposition10. If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\right)$ are two LDFGSs, such that $\breve{\rho'}_i \supseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is postulated as:

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{61}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{G_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{G_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{G_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i) + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{G_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i) + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i).
\end{aligned}\right\} \tag{62}$$

Proof. Follows from Theorems 5 and 8.

Proposition11. If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\right)$ are two LDFGSs, such that $\breve{\rho'}_i \supseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, where $a, b, c, d \in [0, 1]$ are fixed, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{63}$$

where,

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)a + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)b + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)c + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)d + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i).
\end{aligned}\right\} \tag{64}$$

Proof. Analogous to the proof of Theorems 6 and 8.

5 Application: Detection of Road Crimes in Sindh

Although there are many crimes committed on the roads, some are very serious, such as kidnapping, robbery, snatching, and harassment. We may investigate which route in the province of Sindh is more crucial for a specific crime by using an LDFGS. An LDFGS can also tell us which crime is the most chronic and which one is growing rapidly. In addition, we can determine which route is the most crucial for a specific road crime. When creating a policy against a particular crime, the police department may find great guidance and assistance from an LDFGS of road crimes. For instance, an LDFGS for a certain region might be highly beneficial to the police department in combating a crime if an FIR is filed about it. It will also point out which roads are most vulnerable to that specific crime. Consequently, law enforcement can erect checkpoints on particular highways and barricade the vicinity of those roadways.

Consider a set $\mathbb{V}$ of certain cities of province Sindh as:

$$\mathbb{V} = \big\{\text{Sukkur}, \text{Karachi}, \text{Nawabshah}, \text{Dadu}, \text{Larkana}, \text{Khairpur}, \text{Kashmore}, \text{Ghotki}\big\}.$$

Consider $\mathfrak{L}$ be an LDFS on $\mathbb{V}$, as displayed in Table 24.

Table 24. LDFS $\mathfrak{L}$ on $\mathbb{V}$

City	$\big(\langle\varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\rangle, \langle\alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\rangle\big)$
Sukkur	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
Karachi	$\big(\langle 0.9, 0.2\rangle, \langle 0.7, 0.1\rangle\big)$
Nawabshah	$\big(\langle 0.6, 0.1\rangle, \langle 0.6, 0.2\rangle\big)$
Dadu	$\big(\langle 0.5, 0.3\rangle, \langle 0.4, 0.1\rangle\big)$
Larkana	$\big(\langle 0.8, 0.1\rangle, \langle 0.8, 0.2\rangle\big)$
Khairpur	$\big(\langle 0.5, 0.4\rangle, \langle 0.6, 0.3\rangle\big)$
Khashmore	$\big(\langle 0.9, 0.1\rangle, \langle 0.9, 0.1\rangle\big)$
Ghotki	$\big(\langle 0.8, 0.5\rangle, \langle 0.3, 0.4\rangle\big)$

In Table 24, the MD $\varkappa^m_{\mathfrak{L}}(\mathbf{x})$ of a city denotes the presence of crime, while the NMD $\varkappa^n_{\mathfrak{L}}(\mathbf{x})$ signifies non-presence of crime together with reference parameters $\alpha_{\mathfrak{L}}(\mathbf{x})$ = severe crime and $\beta_{\mathfrak{L}}(\mathbf{x})$ = mild crime. In Tables 25-32, we have shown MD, NMD and their corresponding parametric values of various crimes on the road that connect each pair of cities.

Table 25. LDFS of crimes on roads connecting Sukkar with other cities

Crimes	**(Sukkar, Karachi)**	**(Sukkar, Nawabshah)**	**(Sukkar, Dadu)**	**(Sukkar, Larkana)**
Kidnapping	$(\langle 0.7, 0.2\rangle, \langle 0.5, 0.2\rangle)$	$(\langle 0.8, 0.4\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.9, 0.1\rangle, \langle 0.7, 0.1\rangle)$	$(\langle 0.8, 0.3\rangle, \langle 0.5, 0.2\rangle)$
Snatching	$(\langle 0.6, 0.4\rangle, \langle 0.5, 0.3\rangle)$	$(\langle 0.7, 0.5\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.7, 0.2\rangle, \langle 0.7, 0.3\rangle)$	$(\langle 0.4, 0.4\rangle, \langle 0.5, 0.4\rangle)$
Robbery	$(\langle 0.9, 0.4\rangle, \langle 0.3, 0.2\rangle)$	$(\langle 0.2, 0.4\rangle, \langle 0.8, 0.2\rangle)$	$(\langle 0.2, 0.3\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.6, 0.7\rangle, \langle 0.8, 0.1\rangle)$
Harassment	$(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle)$	$(\langle 0.7, 0.3\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.7, 0.3\rangle, \langle 0.5, 0.4\rangle)$

Table 26. LDFS of crimes on roads connecting Nawabshah with other cities

Crimes	**(Nawabshah, Dadu)**	**(Nawabshah, Larkana)**	**(Nawabshah, Khairpur)**	**(Nawabshah, Sukkar)**
Kidnapping	$(\langle 0.7, 0.3\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.9, 0.4\rangle, \langle 0.7, 0.2\rangle)$
Snatching	$(\langle 1, 0.4\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.8, 0.4\rangle, \langle 0.7, 0.2\rangle)$	$(\langle 0.7, 0.4\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle)$
Robbery	$(\langle 0.8, 0.3\rangle, \langle 0.7, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.8, 0.2\rangle, \langle 0.5, 0.3\rangle)$	$(\langle 0.9, 0.4\rangle, \langle 0.8, 0.1\rangle)$
Harassment	$(\langle 0.9, 0.5\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.5, 0.3\rangle)$	$(\langle 0.7, 0.3\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.8, 0.1\rangle)$

Table 27. LDFS of crimes on roads connecting Kashmore with other cities

Crimes	**(Kashmore, Sukkar)**	**(Kashmore, Karachi)**	**(Kashmore, Nawabshah)**	**(Kashmore, Khairur)**
Kidnapping	$(\langle 0.6, 0.5\rangle, \langle 0.5, 0.3\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.2\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.3, 0.2\rangle)$	$(\langle 0.6, 0.3\rangle, \langle 0.3, 0.1\rangle)$
Snatching	$(\langle 0.8, 0.3\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$
Robbery	$(\langle 0.8, 0.6\rangle, \langle 0.4, 0.2\rangle)$	$(\langle 0.9, 0.5\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.5, 0.4\rangle)$
Harassment	$(\langle 0.6, 0.6\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.5, 0.5\rangle, \langle 0.4, 0.2\rangle)$	$(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle)$

Table 28. LDFS of crimes on roads connecting Ghotki with other cities

Crimes	**(Ghotki, Nawabshah)**	**(Ghotki, Sukkar)**	**(Ghotki, Karachi)**	**(Ghotki, Kashmore)**
Kidnapping	$(\langle 0.8, 0.7\rangle, \langle 0.5, 0.2\rangle)$	$(\langle 0.7, 0.5\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.9, 0.6\rangle, \langle 0.6, 0.4\rangle)$
Snatching	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.4\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 08, 0.7\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.5, 0.4\rangle, \langle 0.4, 0.3\rangle)$
Robbery	$(\langle 0.9, 0.7\rangle, \langle 0.6, 0.4\rangle)$	$(\langle 0.6, 0.6\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.8, 0.6\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.4, 0.2\rangle)$
Harassment	$(\langle 0.9, 0.7\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.8, 0.7\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.1\rangle)$	$(\langle 0.9, 0.8\rangle, \langle 0.4, 0.1\rangle)$

Table 29. LDFS of crimes on roads connecting Dadu with other cities

Crimes	**(Dadu, Larkana)**	**(Dadu, Khairpur)**	**(Dadu, Kashmore)**	**(Dadu, Ghotki)**
Kidnapping	$(\langle 0.7, 0.6\rangle, \langle 0.5, 0.3\rangle)$	$(\langle 0.7, 0.5\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.9, 0.4\rangle, \langle 0.4, 0.2\rangle)$	$(\langle 0.8, 0.3\rangle, \langle 0.6, 0.1\rangle)$
Snatching	$(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.5, 0.4\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.8, 0.1\rangle, \langle 0.9, 0.1\rangle)$	$(\langle 0.7, 0.7\rangle, \langle 0.5, 0.1\rangle)$
Robbery	$(\langle 0.6, 0.4\rangle, \langle 0.3, 0.2\rangle)$	$(\langle 0.4, 0.4\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.9, 0.4\rangle, \langle 0.2, 0.3\rangle)$	$(\langle 0.8, 0.4\rangle, \langle 0.3, 0.4\rangle)$
Harassment	$(\langle 0.9, 0.4\rangle, \langle 0.4, 0.4\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.8, 0.2\rangle)$	$(\langle 0.1, 0.4\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.4, 0.7\rangle, \langle 0.2, 0.6\rangle)$

Table 30. LDFS of crimes on roads connecting Khairpur with other cities

Crimes	**(Khairpur, Sukkar)**	**(Khairpur, Kashmore)**	**(Khairpur, Ghotki)**	**(Khairpur, Karachi)**
Kidnapping	$(\langle 0.7, 0.4\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.6, 0.5\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.5, 0.5\rangle)$
Snatching	$(\langle 0.9, 0.4\rangle, \langle 0.3, 0.3\rangle)$	$(\langle 0.7, 0.6\rangle, \langle 0.4, 0.2\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle)$
Robbery	$(\langle 0.7, 0.9\rangle, \langle 0.4, 0.6\rangle)$	$(\langle 0.7, 0.7\rangle, \langle 0.6, 0.3\rangle)$	$(\langle 0.7, 0.8\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.7, 0.5\rangle, \langle 0.6, 0.3\rangle)$
Harassment	$(\langle 0.9, 0.4\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.7, 0.9\rangle, \langle 0.3, 0.5\rangle)$	$(\langle 0.7, 0.6\rangle, \langle 0.4, 0.1\rangle)$	$(\langle 0.8, 0.8\rangle, \langle 0.8, 0.1\rangle)$

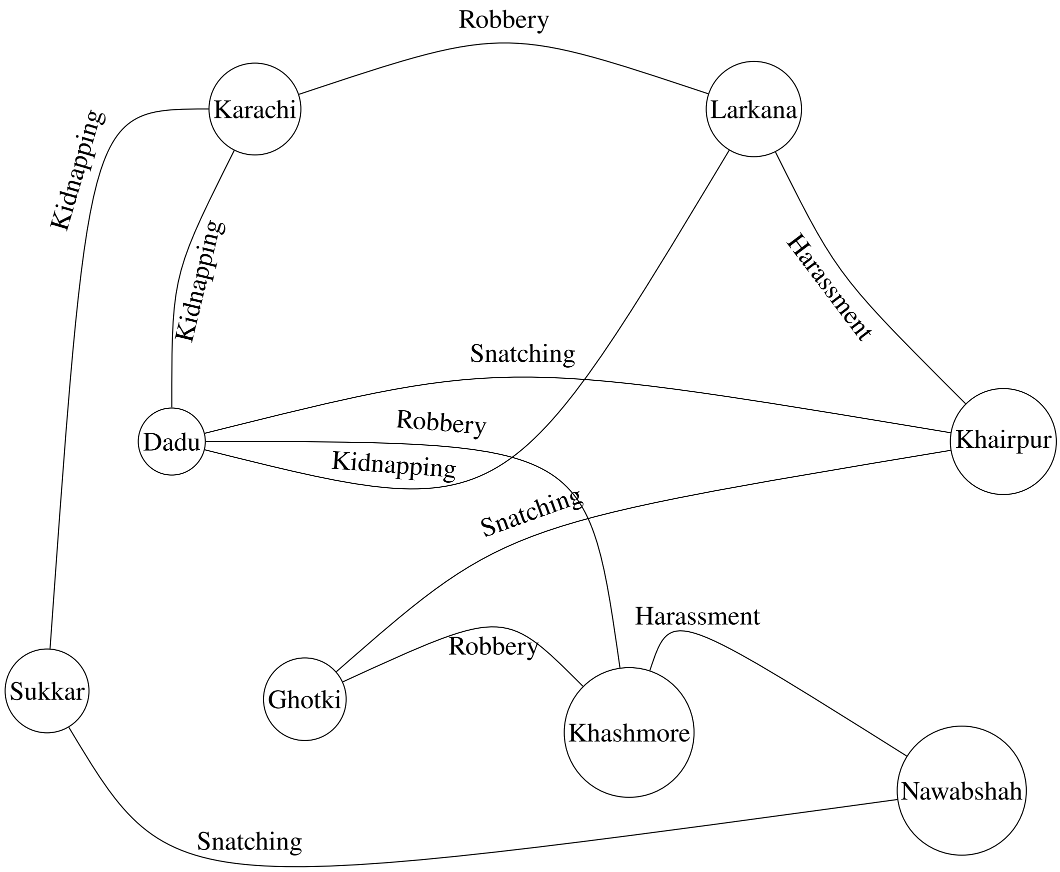

Figure 4. LDFGS indicating the most crucial crimes on roads connecting two cities in Sindh

Table 31. LDFS of crimes on roads connecting Larkana with other cities

Crimes	**(Larkana, Khairpur)**	**(Larkana, Kashmore)**	**(Larkana, Ghotki)**	**(Larkana, Sukkar)**
Kidnapping	$(\langle 0.5, 0.5\rangle, \langle 0.2, 0.2\rangle)$	$(\langle 0.5, 0.8\rangle, \langle 0.4, 0.4\rangle)$	$(\langle 0.7, 0.8\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.4, 0.7\rangle, \langle 0.5, 0.5\rangle)$
Snatching	$(\langle 0.7, 0.4\rangle, \langle 0.5, 0.1\rangle)$	$(\langle 0.2, 0.4\rangle, \langle 0.2, 0.6\rangle)$	$(\langle 0.9, 0.7\rangle, \langle 0.7, 0.3\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.3, 0.3\rangle)$
Robbery	$(\langle 0.5, 0.4\rangle, \langle 0.5, 0.5\rangle)$	$(\langle 0.7, 0.6\rangle, \langle 0.7, 0.1\rangle)$	$(\langle 0.2, 0.4\rangle, \langle 0.3, 0.2\rangle)$	$(\langle 0.5, 0.4\rangle, \langle 0.6, 0.3\rangle)$
Harassment	$(\langle 0.9, 0.2\rangle, \langle 0.6, 0.4\rangle)$	$(\langle 0.7, 0.5\rangle, \langle 0.3, 0.4\rangle)$	$(\langle 0.9, 0.4\rangle, \langle 0.6, 0.2\rangle)$	$(\langle 0.4, 0.4\rangle, \langle 0.1, 0.2\rangle)$

Table 32. LDFS of crimes on roads connecting Karachi with other cities

Crimes	**(Karachi, Larkana)**	**(Karachi, Nawabshah)**	**(Karachi, Dadu)**	**(Karachi, Khairpur)**
Kidnapping	$(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle)$	$(\langle 0.2, 0.4\rangle, \langle 0.4, 0.6\rangle)$	$(\langle 0.7, 0.6\rangle, \langle 0.3, 0.1\rangle)$	$(\langle 0.2, 0.7\rangle, \langle 0.7, 0.2\rangle)$
Snatching	$(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle)$	$(\langle 0.9, 0.9\rangle, \langle 0.9, 0.1\rangle)$	$(\langle 0.4, 0.5\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.6, 0.4\rangle, \langle 0.5, 0.4\rangle)$
Robbery	$(\langle 0.7, 0.5\rangle, \langle 0.7, 0.2\rangle)$	$(\langle 0.8, 0.8\rangle, \langle 0.2, 0.2\rangle)$	$(\langle 0.9, 0.7\rangle, \langle 0.1, 0.3\rangle)$	$(\langle 0.4, 0.5\rangle, \langle 0.6, 0.2\rangle)$
Harassment	$(\langle 0.8, 0.9\rangle, \langle 0.5, 0.4\rangle)$	$(\langle 0.2, 0.3\rangle, \langle 0.4, 0.5\rangle)$	$(\langle 0.6, 0.7\rangle, \langle 0.8, 0.2\rangle)$	$(\langle 0.3, 0.4\rangle, \langle 0.4, 0.6\rangle)$

Different relations can constructed on the set $\mathbb{V}$ as follows: $\rho_1 =$ Kidnapping, $\rho_2 =$ Snatching, $\rho_3 =$ Robbery, $\rho_4 =$ Harassment such that $(\mathbb{V}, \rho_1, \rho_2, \rho_3, \rho_4)$ is a GS. Each element in an arbitrary relation demonstrates a specific crime type that takes place on the road connecting those two cities. Since $(\mathbb{V}, \rho_1, \rho_2, \rho_3, \rho_4)$ is a GS , so each ρ_i are disjoint.

In the light of above information ρ_i are described as follows:

$$\rho_1 = \big\{\text{(Karachi, Dadu), (Dadu, Larkana), (Sukkar, Karachi)}\big\},$$
$$\rho_2 = \big\{\text{(Khaipur, Dadu), (Ghotki, Dadu), (Sukkar, Nawabshah)}\big\},$$
$$\rho_3 = \big\{\text{(Karachi, Larkana), (Khasmore, Dadu), (Ghotki, Kashmore) }\big\},$$
$$\rho_4 = \big\{\text{(Larkana, Khairpure), (Khasmore, Nawabshah) }\big\}.$$

The corresponding LDFSs of ρ_i are given as:

$$\breve{\rho}_1 = \left\{ \begin{array}{l} \big((\text{Karachi, Dadu}), (\langle 0.7, 0.6\rangle, \langle 0.3, 0.1\rangle)\big), \\ \big((\text{Dadu, Larkana}), (\langle 0.7, 0.6\rangle, \langle 0.5, 0.3\rangle)\big), \\ \big((\text{Sukkur, Karachi}), (\langle 0.7, 0.2\rangle, \langle 0.5, 0.2\rangle)\big) \end{array} \right\},$$

$$\breve{\rho}_2 = \left\{ \begin{array}{l} \big((\text{Khairpur, Dadu}), (\langle 0.5, 0.4\rangle, \langle 0.5, 0.4\rangle)\big), \\ \big((\text{Ghotki, Dadu}), (\langle 0.7, 0.7\rangle, \langle 0.5, 0.1\rangle)\big), \\ \big((\text{Sukkur, Nawabshah}), (\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle)\big) \end{array} \right\},$$

$$\breve{\rho}_3 = \left\{ \begin{array}{l} \big((\text{Karachi, Larkana}), (\langle 0.7, 0.5\rangle, \langle 0.7, 0.2\rangle)\big), \\ \big((\text{Khashmore, Dadu}), (\langle 0.9, 0.4\rangle, \langle 0.2, 0.3\rangle)\big), \\ \big((\text{Ghotki, Kashmore}), (\langle 0.8, 0.5\rangle, \langle 0.4, 0.2\rangle)\big) \end{array} \right\},$$

$$\breve{\rho}_4 = \left\{ \begin{array}{l} \big((\text{Larkana, Khairpur}), (\langle 0.9, 0.2\rangle, \langle 0.6, 0.4\rangle)\big), \\ \big((\text{Kashmore, Nawabshah}), (\langle 0.5, 0.5\rangle, \langle 0.4, 0.2\rangle)\big) \end{array} \right\}.$$

Clearly, $(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \breve{\rho}_3, \breve{\rho}_4)$ is an LDFGS which is depicted in Figure 4.

In the light of Figure 4, each edge illustrates the most common crimes that happen on the roadways that link the respective cities. For instance, the most common on roads connecting Karachi and Sukkar is kidnapping. An LDFGS of all cities of the province of Sindh can be very helpful for law-and-order organizations and police departments to circumvent the mentioned road crimes and maintain peace. It would highlight those road crimes that needed to be promptly solved.

6 Discussion and Comparative Analysis

GSs offer an efficient framework for representing scenarios involving multiple relations. Formally, a graph structure consists of n mutually disjoint relations that are both symmetric and irreflexive, allowing for a structured and precise analysis of complex relationships. A multitude of scholars have taken a keen interest in GSs under different fuzzy paradigms. In [18, 19], the authors introduced the concept of FGSs with applications. Sharma and Bansal [20] familiarized the concept of IFGSs. By generalizing the studies reported in [18–20], in this script, we have initiated the notion of LDFGS and several relevant terminologies with concrete illustrations. LDFGSs characterize a noteworthy advancement over traditional FGSs and IFGSs by addressing key limitations associated with MD and NMD in these models. Conventional FGS suffer from a restricted representation of uncertainty, as they only accommodate an MD, which often fails to capture the complexity of real-world problems. IFGS improved upon this by introducing an NMD; however, they still face limitations in handling higher levels of uncertainty and vagueness. In contrast, LDFGS provides a more generalized and robust framework by incorporating reference parameters, permitting a more refined and comprehensive representation of uncertainty. This enhanced flexibility empowers decision-makers by offering a wider range of inputs for both MD and NMD, facilitating more precise modelling of complex relationships. Furthermore, the ability of LDFGS to effectively eliminate restrictive conditions on membership degrees makes them particularly beneficial in decision analysis, network analysis, and optimization problems where uncertainty plays a vital role.

7 Conclusions

GT plays a crucial role in addressing complex problems across diverse domains, including networking, communica -tion, data mining, clustering, image processing, planning, and scheduling. However, in certain scenarios, its classical framework may be insufficient due to intrinsic uncertainties. To address this limitation, FGSs deliver a more effective method for handling uncertainty in complex networks. In this study, we presented the novel concept of LDFGS as an extension of IFGS and LDFG within the broader context of GSs. We established fundamental definitions and concepts in LDFGSs, including $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of a $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, as well as total degree of a vertex in an LDFGS. Furthermore, we introduced the notions of $\breve{\rho}_i$-size, size, and order of an LDFGS. Further, we explored advanced structural properties by defining and analyzing the maximal product of two LDFGSs, strong LDFGSs, and the degree and $\breve{\rho}_i$-degree of the maximal product. To demonstrate the practical applicability of LDFGSs, we presented a real-world case study on crime analysis, identifying the most critical crime type across various cities in the Sindh province.

One of the key limitations of LDFGSs is their incapability to incorporate a neutral degree. In numerous real-world applications, especially in decision-making situations involving uncertainty and vagueness, the inclusion of a neutral degree is indispensable to capture indecisiveness or partial agreement. While LDFGSs deliver a more flexible framework compared to traditional FGSs, their binary approach to uncertainty (considering only MD and NMD) confines their applicability in cases where a neutral degree is necessary. For future study, an extension of LDFGS could be developed by integrating a neutral degree, such as Spherical linear Diophantine fuzzy GSs and Spherical linear Diophantine fuzzy soft GSs. In addition, exploring hybrid models that combine LDFGS with other uncertainty-handling frameworks could further improve their applicability in realms like social network analysis, medical diagnosis, and multi-criteria decision-making.

Data Availability

Not applicable.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] K. T. Atanassov, "Intuitionistic Fuzzy Sets," in *Intuitionistic Fuzzy Sets: Theory and Applications*. Heidelberg, Germany: Physica-Verlag HD, 1999, pp. 1–137. https://doi.org/10.1007/978-3-7908-1870-3_1

[3] R. R. Yager, "Pythagorean fuzzy subsets," in *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, Edmonton, AB, Canada, 2013, pp. 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[4] R. R. Yager, "Generalized orthopair fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1222–1230, 2016. https://doi.org/10.1109/TFUZZ.2016.2604005

[5] M. Riaz and M. R. Hashmi, "Linear diophantine fuzzy set and its applications towards multi-attribute decision-making problems," *J. Intell. Fuzzy Syst.*, vol. 37, no. 4, pp. 5417–5439, 2019. https://doi.org/10.3233/JIFS-190550

[6] S. Ayub, M. Shabir, M. Riaz, M. Aslam, and R. Chinram, "Linear diophantine fuzzy relations and their algebraic properties with decision making," *Symmetry*, vol. 13, no. 6, p. 945, 2021. https://doi.org/10.3390/sym13060945

[7] S. Ayub, M. Shabir, M. Riaz, F. Karaaslan, D. Marinkovic, and D. Vranjes, "Linear diophantine fuzzy rough sets on paired universes with multi stage decision analysis," *Axioms*, vol. 11, no. 12, p. 686, 2022. https://doi.org/10.3390/axioms11120686

[8] S. Ayub, M. Shabir, M. Riaz, W. Mahmood, D. Bozanic, and D. Marinkovic, "Linear diophantine fuzzy rough sets: A new rough set approach with decision making," *Symmetry*, vol. 14, no. 3, p. 525, 2022. https://doi.org/10.3390/sym14030525

[9] S. Ayub, M. Shabir, and R. Gul, "Another approach to linear diophantine fuzzy rough sets on two universes and its application towards decision-making problems," *Phys. Scr.*, vol. 98, no. 10, p. 105240, 2023. https://doi.org/10.1088/1402-4896/acf811

[10] H. Kamacı, "Linear diophantine fuzzy algebraic structures," *J. Ambient Intell. Humaniz. Comput.*, vol. 12, no. 11, pp. 10 353–10 373, 2021. https://doi.org/10.1007/s12652-020-02826-x

[11] A. Rosenfeld, "Fuzzy graphs," in *Proceedings of the US–Japan Seminar on Fuzzy Sets and their Applications*, Berkeley, CA, USA, 1975, pp. 77–95. https://doi.org/10.1016/B978-0-12-775260-0.50008-6

[12] M. G. Voskoglou and T. Pramanik, "Fuzzy Graphs and Fuzzy Hypergraphs," in *Handbook of Research on Advanced Applications of Graph Theory in Modern Society*. Hershey, PA, USA: IGI Global, 2020, pp. 437–468. https://doi.org/10.4018/978-1-5225-9380-5.ch019

[13] R. Parvathi, M. G. Karunambigai, and K. T. Atanassov, "Operations on intuitionistic fuzzy graphs," in *2009 IEEE International Conference on Fuzzy Systems*, Jeju, Korea (South), 2009, pp. 1396–1401. https://doi.org/10.1109/FUZZY.2009.5277067

[14] M. Akram, S. Alsulami, F. Karaaslan, and A. Khan, "q-Rung orthopair fuzzy graphs under Hamacher operators," *J. Intell. Fuzzy Syst.*, vol. 40, no. 1, pp. 1367–1390, 2021. https://doi.org/10.3233/JIFS-201700

[15] M. Akram, J. M. Dar, and S. Naz, "Certain graphs under pythagorean fuzzy environment," *Complex Intell. Syst.*, vol. 5, pp. 127–144, 2019. https://doi.org/10.1007/s40747-018-0089-5

[16] M. Z. Hanif, N. Yaqoob, M. Riaz, and M. Aslam, "Linear diophantine fuzzy graphs with new decision-making approach," *AIMS Math*, vol. 7, no. 8, pp. 14 532–14 556, 2022. https://doi.org/10.3934/math.2022801

[17] E. Sampathkumar, "Generalized graph structures," *Bull. Kerala Math. Assoc.*, vol. 3, no. 2, pp. 67–123, 2006.

[18] M. Akram and M. Sitara, "Certain fuzzy graph structures," *J. Appl. Math. Comput.*, vol. 61, pp. 25–56, 2019. https://doi.org/10.1007/s12190-019-01237-2

[19] M. Sitara, M. Akram, and M. Yousaf Bhatti, "Fuzzy graph structures with application," *Mathematics*, vol. 7, no. 1, p. 63, 2019. https://doi.org/10.3390/math7010063

[20] P. K. Sharma and V. Bansal, "On intuitionistic fuzzy graph structures," *IOSR J. Math.*, vol. 12, no. 5, pp. 28–33, 1982. https://doi.org/10.9790/5728-1205012833

[21] M. Sitara, M. Akram, and M. Riaz, "Decision-making analysis based on q-rung picture fuzzy graph structures," *J. Appl. Math. Comput.*, vol. 67, no. 1, pp. 541–577, 2021. https://doi.org/10.1007/s12190-020-01471-z

[22] H. Bustince, "Construction of intuitionistic fuzzy relations with predetermined properties," *Fuzzy Sets Syst.*, vol. 109, no. 3, pp. 379–403, 2000. https://doi.org/10.1016/S0165-0114(97)00381-3

[23] T. Dinesh and T. V. Ramakrishnan, "On generalised fuzzy graph structures," *Appl. Math. Sci.*, vol. 5, no. 4, pp. 173–180, 2011.

[24] P. K. Sharma, V. Bansal, and D. S. Pathania, "Regular intuitionistic fuzzy graph structure," *Int. J. Res. Appl. Sci. Eng. Technol.*, vol. 6, no. 4, pp. 3722–3729, 2018. https://doi.org/10.22214/IJRASET.2018.4619