Acadlore Transactions on Applied Mathematics and Statistics
https://www.acadlore.com/journals/ATAMS

Mathematical Model to Evaluate of Client Behavior in the Automobile Sector Based on T-Spherical Fuzzy Information within Dombi Environment

Muhammad Altaf Khan[1,2,3], Khaista Rahman[4*]

[1] School of Health and Environmental Science, Korea University, 02841 Seoul, South Korea
[2] Faculty of Natural and Agricultural Sciences, University of the Free State, 9300 Bleomfontein, South Africa
[3] Laboratory of Vaccine and Biomolecules, Institute of Bioscience, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia
[4] Department of Mathematics, Shaheed Benazir Bhutto University Sheringal, 1800 Dir Upper, Pakistan

[*] Correspondence: Khaista Rahman (khaista@sbbu.edu.pk)

Received: 01-25-2025 **Revised:** 03-10-2025 **Accepted:** 03-23-2025

Citation: M. A. Khan, K. Rahman, "Mathematical model to evaluate of client behavior in the automobile sector based on T-spherical fuzzy information within Dombi environment," *Acadlore Trans. Appl Math. Stat.*, vol. 3, no. 1, pp. 52–64, 2025. https://doi.org/10.56578/atams030104.

Abstract: Automobiles play a vital role in daily life, providing suitable and efficient transportation for work, school, and errands. They also support essential services like emergency response, goods delivery, and public transportation systems. This increased variety means that car manufacturers are competing intensely to attract customers and maximize their profits. However, making the right choice when buying a car can be challenging due to the wide range of factors to consider. This study introduces a new approach that uses Dombi operators combined with T-spherical fuzzy numbers (T-SFNs) to help improve the decision-making process. This method reduces the uncertainty and imprecision that often comes with decision-making, especially when selecting a car. The aim is to help customers make better, more informed choices and avoid financial difficulties. To achieve this, the study develops several innovative operators namely T-spherical fuzzy Dombi weighted averaging (T-SFDWA), T-spherical fuzzy Dombi ordered weighted averaging (T-SFDOWA), T-spherical fuzzy Dombi weighted geometric (T-SFDWG), T-spherical fuzzy Dombi ordered weighted geometric (T-SFDOWG). These methods offer flexibility, suppleness and can adapt to real-world problems where factors are constantly changing. By managing uncertainty and hesitation effectively, these approaches help decision-makers evaluate complex situations with multiple variables. A practical example, such as choosing a car, demonstrates how these approaches can evaluate important criteria like price, safety, and fuel efficiency. Ultimately, these methods ensure that consumers can make the best decision, even in uncertain and complex situations.

Keywords: Automobile selection; T-SF-Model; Dombi approaches; Decision-making

1 Introduction

Automobiles are essential in our daily lives, making travel faster and easier. They help us get to work, visit family, or go on vacations, saving time and effort. Cars, buses, and trucks are also important for transporting goods, helping businesses run smoothly. In emergencies, vehicles like ambulances and fire trucks respond quickly, saving lives. Automobiles connect people, making it easier to travel long distances and explore new places. They have improved both personal mobility and global trade, allowing goods to be exchanged worldwide. However, choosing the right car can be tough because there are so many different options available. Factors like price, fuel efficiency, and safety features need to be considered. In the end, the right automobile can make life much more convenient and enjoyable. Automobiles have truly changed the way we live and interact with the world.

Choosing the more suitable automobile can be difficult because there are many different types with varying features. To handle this complexity, Zadeh [1] introduced the concept of fuzzy sets (FSs), which allow us to consider and compare cars based on subjective qualities like comfort or style. FSs help make decisions easier by recognizing that not all factors can be measured precisely. Following the inception of FS theory, researchers have significantly broadened its scope by introducing various extensions aimed at enhancing its capabilities. These extensions include inventive concepts like intuitionistic fuzzy sets (IFSs) [2], which adeptly capture uncertainty and hesitation in a more nuanced manner. IFSs expand upon FSs to offer a more inclusive model that effectively addresses uncertainty and vagueness in various applications. Unlike FSs, IFSs not only include a degree of membership but also introduce a separate degree of non-membership, with their combined sum being less than or equal to one. This extension provides a comprehensive approach to handling imprecision and ambiguity in different contexts, making it particularly useful in decision-making processes where uncertainty about both membership and non-membership exists simultaneously. Later on, several researchers introduced various aggregation operators. These operators serve as crucial tools for effectively combining and harmonizing data, allowing for a more comprehensive and coherent understanding of the underlying information. Noteworthy contributions have been made by researchers such as Wang and Liu [3], Wang and Liu [4], Xu [5], Xu and Yager [6], Rahman et al. [7], Rahman [8] who have significantly advanced the field of operators by incorporating intuitionistic fuzzy numbers. Their work introduces various techniques specifically designed to handle uncertainties and imprecisions in decision-making scenarios. Additionally, Xia et al. [9] and Yu [10] have put forth aggregation operators that operate within predefined confidence levels. Dombi [11], in his research, has proposed the Dombi operational laws as an alternative to traditional algebraic and Einstein operational laws. These laws offer a potentially novel and effective framework for certain operations, presenting advantages over the conventional algebraic and Einstein approaches. Seikh and Mandal [12] have contributed to the field by introducing Dombi aggregation operators (DAOs), which are fuzzy logic concepts widely applied in decision-making and various domains. These operators blend the notion of IFSs with Dombi operators to address decision-making process that involve fuzziness and vagueness. Figure 1 shows a visual diagram of the IFS system, making it easier to understand how its parts work together.

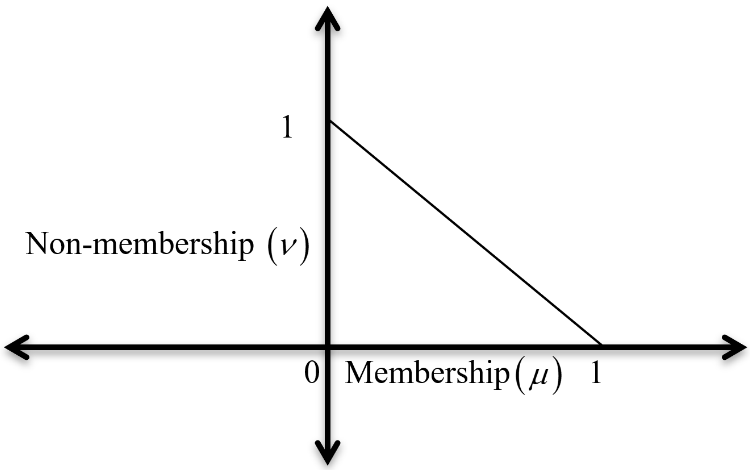

Figure 1. Graphical presentation of IFS

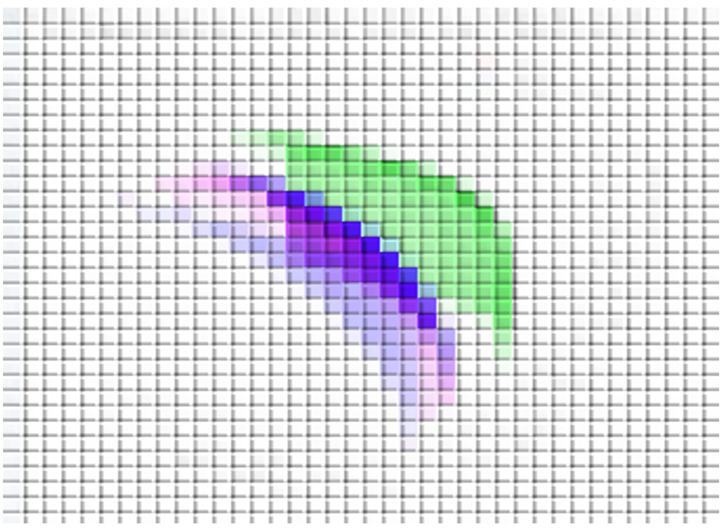

Figure 2. Space comparison of PFSs, SFSs, T-SFSs

Later on, Cường [13] introduced picture fuzzy sets (PFSs), which can be seen as an expansion of traditional FSs. The primary objective of PFSs is to address and encompass intricate and organized uncertainties by integrating detailed pixel-level information. This concept allows for a more detailed demonstration of vagueness, especially in situations where spatial information or visual data is involved. So, in PFSs each element can be presented mathematically as: (μ, η, ν), where μ stands for membership, η stands for neutral membership, and ν stands for non-membership under condition $\mu + \eta + \nu \leq 1$. Ashraf et al. [14] developed spherical fuzzy sets (SFSs). This innovation was introduced to overcome the constraints associated with PcFSs $\mu + \eta + \nu \leq 1$ to $\mu^2 + \eta^2 + \nu^2 \leq 1$. In essence, SFSs provide a more flexible and versatile framework, offering relaxation from the limitations posed by PFSs. Mahmood et al. [15] brought forth the concept of T-spherical fuzzy sets (T-FSs) as advancement over the limitations posed by SpFSs $\mu^2 + \eta^2 + \nu^2 \leq 1$ to $\mu^t + \eta^t + \nu^t \leq 1$. This innovation serves to broaden the scope and flexibility of FSs, allowing for a more versatile and adaptable approach in various applications. T-FSs offer a fresh perspective on handling uncertainty and vagueness, diverging from conventional FS theories. These advancements introduce more adaptable and geometrically intuitive ways of representing information, enabling a more comprehensive depiction of relationships in real-world situations. T-FSs provide alternative methods that enhance our ability to model complex and nuanced uncertainties in various scenarios. Figure 2 shows the comparison of different models.

Building on the advantages of existing approaches, this study introduces DAOs within the framework of T-spherical fuzzy numbers (T-SFNs). We establish key laws and propose approaches based on these principles. Finally, we demonstrate their applicability through a decision-making analysis for selecting the optimal automobile.

This research is analytically organized to ensure a clear and comprehensive understanding of the study topic. Section 2 presents some existing models relevant to the research, establishing a foundational base for further discussion. Section 3 provides an in-depth look into Dombi norms, highlighting their basic principles. Section 4 explores Dombi-based methods, specifically T-spherical fuzzy Dombi weighted averaging (T-SFDWA), T-spherical fuzzy Dombi ordered weighted averaging (T-SFDOWA), T-spherical fuzzy Dombi weighted geometric(T-SFDWG), and T-spherical fuzzy Dombi ordered weighted geometric (T-SFDOWG), forming the core of the new proposed methodology. In Section 5 real world applications. Section 6 presents a practical example to demonstrate the real-world applicability of the proposed methods. Section 7 conducts a sensitivity analysis to assess the robustness of the approach. Section 8 offers a comparative analysis with existing models to highlight the advantages of the proposed methods. Section 9 showcases a practical application of the developed models, emphasizing their utility. Finally, Section 10 concludes the paper by summarizing key findings and providing meaningful insights for future research.

2 Basic Existing Work

This unit explains the key ideas including PFS, SFS, T-SFS, which are important for our new upcoming research. This concept will help us understand and analyze specific data in our study. Figure 3 shows the T-SFS along with its special cases, showcasing the generalization and relationships among different FS models.

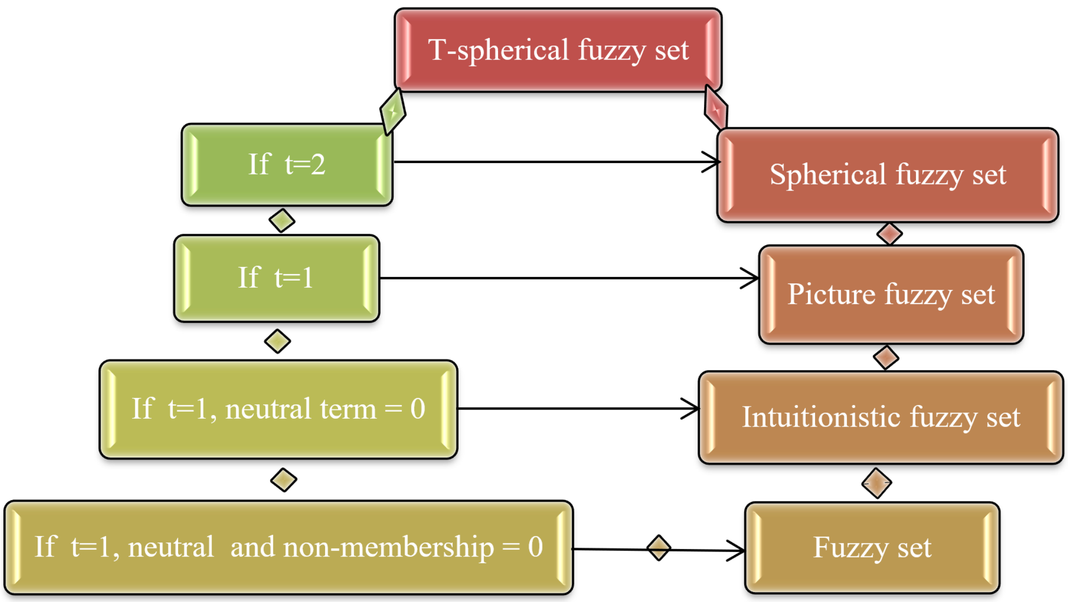

Figure 3. T-SFS and their special cases

Definition 1. A PFS is defined as: $P = \{z, \langle\mu, \eta, \nu\rangle | z \in Z\}$, where $\mu : Z \to [0,1]$, $\eta : Z \to [0,1]$, $\nu : Z \to [0,1]$ are called membership, neutral membership, and non-membership under condition $\mu + \eta + \nu \leq 1$, respectively [13].

Definition 2. A SFS is mathematically defined as: $S = \{z, \langle\mu, \eta, \nu\rangle | z \in Z\}$, where $\mu : Z \to [0,1]$, $\eta : Z \to [0,1]$, $\nu : Z \to [0,1]$ are called membership, neutral membership, and non-membership respectively with $\mu^2 + \eta^2 + \nu^2 \leq 1$ [14].

Definition 3. A T-SF is defined as: $T = \{z, \langle\mu, \eta, \nu\rangle | z \in Z\}$, where $\mu : Z \to [0,1]$, $\eta : Z \to [0,1]$, $\nu : Z \to [0,1]$ are called membership, neutral membership, and non-membership respectively with $\mu^t + \eta^t + \nu^t \leq 1$ and t be any positive number [15].

Definition 4. Let $\mathbb{C} = (\mu, \eta, \nu)$ be T-SFN, then its score functions can be defined as: $scor(\mathbb{C}) = \frac{1+\mu^t+\eta^t-v^t}{3}$ with condition: $scor(\mathbb{C}) \in [-2, 2]$ [15].

3 Basic Laws Based on Dombi Norms

In this section, we introduce basic operational laws using Dombi norms to build different fuzzy aggregation operators. These laws help combine uncertain or imprecise information in a consistent and easy-to-understand way. By following these rules, we make sure decisions are logical and reliable, even when the input values vary. Figure 4 shows the structured procedure of the new proposed research study.

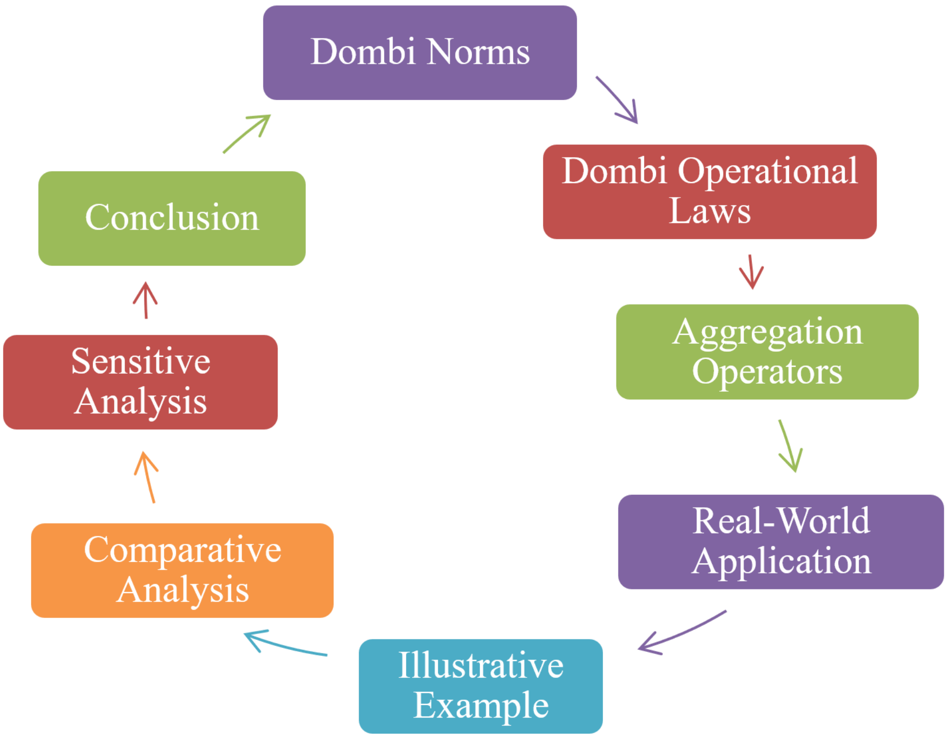

Figure 4. The designed procedure of the proposed study

Definition 5. Let r and s be natural numbers. The Dombi norms are defined as follows:

$$T_D(r, s) = \frac{1}{1 + \left(\left(\frac{1-r}{r}\right)^m + \left(\frac{1-s}{s}\right)^m\right)^{\frac{1}{m}}} \tag{1}$$

$$S_D(r, s) = 1 - \frac{1}{1 + \left(\left(\frac{r}{1-r}\right)^m + \left(\frac{s}{1-s}\right)^m\right)^{\frac{1}{m}}} \tag{2}$$

The Dombi t-norm (T) and t-conorm (S) are mathematical operations, where the parameter m ($m \geq 1$) controls their shape. This parameter allows flexibility in how these operations behave. As m increases, t-norm going to be the minimum operation, while t-conorm approaches to be the maximum operation.

Definition 6. Let $\mathtt{C}_j = (\mu_j, \eta_j, \nu_j)\,(1 \le j \le 3)$ are T-SFNs, and $\hbar \succ 0$, then

(i)

$$\mathtt{C}_1 \oplus \mathtt{C}_2 = \left(\begin{array}{l} \sqrt[t]{1 - \frac{1}{1 + \left[\left(\frac{\mu_1^t}{1-\mu_1^t} \right)^m + \left(\frac{\mu_2^t}{1-\mu_2^t} \right)^m \right]^{1/m}}}, \quad \sqrt[t]{\frac{1}{1 + \left[\left(\frac{1-\eta_1^t}{\eta_1^t} \right)^m + \left(\frac{1-\eta_2^t}{\eta_2^t} \right)^m \right]^{1/m}}}, \\ \sqrt[t]{\frac{1}{1 + \left[\left(\frac{1-\nu_1^t}{\nu_1^t} \right)^m + \left(\frac{1-\nu_2^t}{\nu_2^t} \right)^m \right]^{1/m}}} \end{array} \right)$$

(ii)

$$\mathtt{C}_1 \otimes \mathtt{C}_2 = \left(\begin{array}{l} \sqrt[t]{\frac{1}{1 + \left[\left(\frac{1-\mu_1^t}{\mu_1^t} \right)^m + \left(\frac{1-\mu_2^t}{\mu_2^t} \right)^m \right]^{1/m}}}, \quad \sqrt[t]{1 - \frac{1}{1 + \left[\left(\frac{\eta_1^t}{1-\eta_1^t} \right)^m + \left(\frac{\eta_2^t}{1-\eta_2^t} \right)^m \right]^{1/m}}}, \\ \sqrt[t]{1 - \frac{1}{1 + \left[\left(\frac{\nu_1^t}{1-\nu_1^t} \right)^m + \left(\frac{\nu_2^t}{1-\nu_2^t} \right)^m \right]^{1/m}}} \end{array} \right)$$

(iii)

$$\hbar(\mathtt{C}) = \left(\sqrt[t]{1 - \frac{1}{1 + \left(\hbar \left(\frac{\mu^t}{1-\mu^t} \right)^m \right)^{1/m}}}, \sqrt[t]{\frac{1}{1 + \left(\hbar \left(\frac{1-\eta^t}{\eta^t} \right)^m \right)^{1/m}}}, \sqrt[t]{\frac{1}{1 + \left(\hbar \left(\frac{1-\nu^t}{\nu^t} \right)^m \right)^{1/m}}} \right)$$

(iv)

$$(\mathtt{C})^{\hbar} = \left(\sqrt[t]{\frac{1}{1 + \left(\hbar \left(\frac{1-\mu^t}{\mu^t} \right)^m \right)^{1/m}}}, \sqrt[t]{1 - \frac{1}{1 + \left(\hbar \left(\frac{\eta^t}{1-\eta^t} \right)^m \right)^{1/m}}}, \sqrt[t]{1 - \frac{1}{1 + \left(\hbar \left(\frac{\nu^t}{1-\nu^t} \right)^m \right)^{1/m}}} \right)$$

Definition 7. Let $\mathtt{C}_j\ (j = 1, 2)$ are two T-SFNs, then

(i) $\mathtt{C}_1 \cup \mathtt{C}_2 = \{ (z, T(\mu_1(z), \mu_2(z)), S(\eta_1(z), \eta_2(z)), S(\nu_1(z), \nu_2(z))) | \, z \in Z \}$

(ii) $\mathtt{C}_1 \cap \mathtt{C}_2 = \{ (z, S(\mu_1(z), \mu_2(z)), S(\eta_1(z), \eta_2(z))), T(\nu_1(z), \nu_2(z)) | \, z \in Z \}$

(iii) Complement: $\mathtt{C}^c = (\nu, \eta, \mu)$

Theorem 1. Let $\mathtt{C}_j\ (1 \le j \le 3)$ are three T-SFNs, then

(i) $\mathtt{C}_1 \cup \mathtt{C}_2 = \mathtt{C}_2 \cup \mathtt{C}_1$

(ii) $\mathtt{C}_1 \cap \mathtt{C}_2 = \mathtt{C}_2 \cap \mathtt{C}_1$

(iii) $\mathtt{C}_1 \cup \mathtt{C}_2 \cup \mathtt{C}_3 = \mathtt{C}_1 \cup \mathtt{C}_3 \cup \mathtt{C}_2$

(iv) $\mathtt{C}_1 \cap \mathtt{C}_2 \cap \mathtt{C}_3 = \mathtt{C}_1 \cap \mathtt{C}_3 \cap \mathtt{C}_2$

Proof. We can prove the given Theorem by using Definition 7.

(i) Since $\mathtt{C}_1$ and $\mathtt{C}_2$ are two CT-SFNs, then by Definition 7, we have

$$\begin{aligned} \mathtt{C}_1 \cup \mathtt{C}_2 &= \big(\max\{\mu_1, \mu_2\},\ \min\{\eta_1, \eta_2\},\ \min\{\nu_1, \nu_2\} \big) \\ &= \big(\max\{\mu_2, \mu_1\},\ \min\{\eta_2, \eta_1\},\ \min\{\nu_2, \nu_1\} \big) \\ &= \mathtt{C}_2 \cup \mathtt{C}_1 \end{aligned}$$

(ii) Again, by using Definition 7, we have

$$\begin{aligned} \mathtt{C}_1 \cap \mathtt{C}_2 &= \big(\min\{\mu_1, \mu_2\},\ \max\{\eta_1, \eta_2\},\ \max\{\nu_1, \nu_2\} \big) \\ &= \big(\min\{\mu_2, \mu_1\},\ \max\{\eta_2, \eta_1\},\ \max\{\nu_2, \nu_1\} \big) \\ &= \mathtt{C}_2 \cap \mathtt{C}_1 \end{aligned}$$

(iii) By using Definition 7, we have

$$\begin{aligned} (\mathtt{C}_1 \cup \mathtt{C}_2) \cup \mathtt{C}_3 &= \big(\max\{\mu_1, \mu_2\},\ \min\{\eta_1, \eta_2\},\ \min\{\nu_1, \nu_2\} \big) \cup \mathtt{C}_3 \\ &= \big(\max\{\mu_1, \mu_2, \mu_3\},\ \min\{\eta_1, \eta_2, \eta_3\},\ \min\{\nu_1, \nu_2, \nu_3\} \big) \\ &= \big(\max\{\mu_1, \mu_3, \mu_2\},\ \min\{\eta_1, \eta_3, \eta_2\},\ \min\{\nu_1, \nu_3, \nu_2\} \big) \\ &= (\mathtt{C}_1 \cup \mathtt{C}_3) \cup \mathtt{C}_2 \end{aligned}$$

(iv) By using Definition 7, we have

$$\begin{aligned}
(\mathtt{C}_1 \cap \mathtt{C}_2) \cap \mathtt{C}_3 &= \big(\min\{\mu_1, \mu_2\},\ \max\{\eta_1, \eta_2\},\ \max\{\nu_1, \nu_2\}\big) \cap \mathtt{C}_3 \\
&= \big(\min\{\mu_1, \mu_2, \mu_3\},\ \max\{\eta_1, \eta_2, \eta_3\},\ \max\{\nu_1, \nu_2, \nu_3\}\big) \\
&= \big(\min\{\mu_1, \mu_3, \mu_2\},\ \max\{\eta_1, \eta_3, \eta_2\},\ \max\{\nu_1, \nu_3, \nu_2\}\big) \\
&= \mathtt{C}_1 \cap \mathtt{C}_3 \cap \mathtt{C}_2
\end{aligned}$$

4 Dombi Aggregation Operators (DAOs)

In FS theory, DAOs help manage unclear or uncertain information, improving decision-making analysis. These approaches handle complex data more effectively. The new operators, like such as T-SFDWA, T-SFDOWA, T-SFDWG, T-SFDOWG, allow for better combining of data in uncertain situations. These methods have important features namely monotonicity, boundedness and idempotency, meaning repeating the process gives the same outcome. They can handle complex relationships, making them useful in difficult decision-making situations.

Definition 8. Let $\mathtt{C}_j$ $(1 \le j \le n)$ be a group of T-SFNs, and $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ be their weighted vectors with limitation $\kappa_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} \kappa_j = 1$, then the T-SFDWA operator can be defined as:

$$\text{T-SFDWA}_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big) = \left(\begin{array}{l} \sqrt[t]{1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{\mu_j^t}{1-\mu_j^t}\right)^m\right)^{1/m}}}, \quad \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\eta_j^t}{\eta_j^t}\right)^m\right)^{1/m}}}, \\ \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\nu_j^t}{\nu_j^t}\right)^m\right)^{1/m}}} \end{array} \right)$$

Idempotency: Let $\mathtt{C}_j = (\mu_j, \eta_j, \nu_j)$ $(1 \le j \le n)$ are T-SFNs and $\mathtt{C}_j = \mathtt{C}$ for all j then

$$\text{T-SFDWA}_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big) = \kappa_1\, \mathtt{C}_1 \oplus \kappa_2\, \mathtt{C}_2 \oplus \cdots \oplus \kappa_n\, \mathtt{C}_n = \mathtt{C} \tag{3}$$

Boundedness: Let $\mathtt{C}_j(1 \le j \le n)$ are T-SFNs, $\mathtt{C}_{\max} = \max\big\{\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big\} = (\mu_{\max}, \eta_{\max}, \upsilon_{\max})$ and $\mathtt{C}_{\min} = \min\big\{\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big\} = (\mu_{\min}, \eta_{\min}, \upsilon_{\min})$, where $\mu_{\max} = \max\limits_j \{\mu_j\}$, $\eta_{\max} = \max\limits_j \{\hbar_j\}$, $\nu_{\max} = \min\limits_j \{\nu_j\}$, $\mu_{\min} = \min\limits_j \{\mu_j\}$, $\hbar_{\min} = \min\limits_j \{\hbar_j\}$, $\nu_{\min} = \min\limits_j \{\nu_j\}$ then

$$\mathtt{C}_{\min} \le T\text{-SFDW}A_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big) \le \mathtt{C}_{\max} \tag{4}$$

Monotonicity: Let there are two different families of finite T-SFNs, such that $\mathtt{C}_j$ $(1 \le j \le n)$, $\mathtt{C}_j^\circ$ $(1 \le j \le n)$ for all j satisfying the conditions: $\mu_j \le \mu_j^\circ$, $\eta_j \le \eta_j^\circ$, $\nu_j \ge \nu_j^\circ$, then

$$\text{T-SFDW}A_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \mathtt{C}_3, \ldots, \mathtt{C}_n\big) \le \text{T-SFDWA}_\kappa\big(\mathtt{C}_1^\circ, \mathtt{C}_2^\circ, \mathtt{C}_3^\circ, \ldots, \mathtt{C}_n^\circ\big) \tag{5}$$

Definition 9. Let $\mathtt{C}_j(1 \le j \le n)$ be a group of T-SFNs, $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ be their weights with conditions: $\kappa_j \in [0, 1]$, $\sum_{j=1}^{n} \kappa_j = 1$, and $(\partial(1), \partial(2), \ldots, \partial(n))$ be any rearrangement of $(1, 2, \ldots, n)$ with $\mathtt{C}_{\partial(j)} \le \mathtt{C}_{\partial(j-1)}$, then T-SFDOWA operator is defined mathematically as:

$$\text{T-SFDOWA}_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big) = \left(\begin{array}{l} \sqrt[t]{1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{\mu_{\partial(j)}^t}{1-\mu_{\partial(j)}^t}\right)^m\right)^{1/m}}}, \quad \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\eta_{\partial(j)}^t}{\eta_{\partial(j)}^t}\right)^m\right)^{1/m}}}, \\ \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\nu_{\partial(j)}^t}{\nu_{\partial(j)}^t}\right)^m\right)^{1/m}}} \end{array} \right)$$

Definition 10. The T-SFDWG operator can be written mathematically as:

$$\text{T-SFDWG}_\kappa\big(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\big) = \left(\begin{array}{l} \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\mu_j^t}{\mu_j^t}\right)^m\right)^{1/m}}}, \quad \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\eta_j^t}{\eta_j^t}\right)^m\right)^{1/m}}}, \\ \sqrt[t]{1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{\nu_j^t}{1-\nu_j^t}\right)^m\right)^{1/m}}} \end{array} \right)$$

where, $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ be their weighted vectors with $\kappa_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} \kappa_j = 1$.

Idempotency: Let $\mathtt{C}_j = (\mu_j, \eta_j, \nu_j)\ (1 \le j \le n)$ are T-SFNs and $\mathtt{C}_j = \mathtt{C}$ for all j then

$$\text{T-SFDWG}_{\kappa}\left(\mathtt{C}_1, \mathtt{C}_2, ..., \mathtt{C}_n\right) = \left(\mathtt{C}_1\right)^{\kappa_1} \otimes \left(\mathtt{C}_2\right)^{\kappa_2} \otimes ... \otimes \left(\mathtt{C}_n\right)^{\kappa_n} = \mathtt{C} \tag{6}$$

Boundedness: Let $\mathtt{C}_j\ (1 \le j \le n)$ are T-SFNs, then

$$\mathtt{C}_{\min} \le \text{T-SFDWA}_{\kappa}\left(\mathtt{C}_1, \mathtt{C}_2, \mathtt{C}_3, ..., \mathtt{C}_n\right) \le \mathtt{C}_{\max} \tag{7}$$

where, $\mathtt{C}_{\max} = (\mu_{\max}, \eta_{\max}, \upsilon_{\max})$ and $\mathtt{C}_{\min} = (\mu_{\min}, \eta_{\min}, \upsilon_{\min})$ with $\mu_{\max} = \max\limits_j \{\mu_j\}$, $\eta_{\max} = \max\limits_j \{\hbar_j\}$, $\nu_{\max} = \min\limits_j \{\nu_j\}$, $\mu_{\min} = \min\limits_j \{\mu_j\}$, $\hbar_{\min} = \min\limits_j \{\hbar_j\}$, $\nu_{\min} = \min\limits_j \{\nu_j\}$.

Monotonicity: Let there are two different families of finite T-SFNs, such that $\mathtt{C}_j\ (1 \le j \le n)$, $\mathtt{C}_j^{\circ}\ (1 \le j \le n)$ for all j satisfying the conditions: $\mu_j \le \mu_j^{\circ}, \eta_j \le \eta_j^{\circ}, \nu_j \ge \nu_j^{\circ}$, then

$$\text{T-SFDWG}_{\kappa}\left(\mathtt{C}_1, \mathtt{C}_2, \mathtt{C}_3, ..., \mathtt{C}_n\right) \le \text{T-SFDWG}_{\kappa}\left(\mathtt{C}_1^{\circ}, \mathtt{C}_2^{\circ}, \mathtt{C}_3^{\circ}, ..., \mathtt{C}_n^{\circ}\right) \tag{8}$$

Definition 11. The T-SFDOWG operator can be defined as:

$$\text{T-SFDOWG}_{\kappa}\left(\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_n\right) = \left(\begin{array}{c} \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\mu_{\partial(j)}^t}{\mu_{\partial(j)}^t}\right)^m\right)^{1/m}}}, \quad \sqrt[t]{\dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{1-\eta_{\partial(j)}^t}{\eta_{\partial(j)}^t}\right)^m\right)^{1/m}}}, \\ \sqrt[t]{1 - \dfrac{1}{1 + \left(\sum_{j=1}^{n} \kappa_j \left(\frac{\nu_{\partial(j)}^t}{1-\nu_{\partial(j)}^t}\right)^m\right)^{1/m}}} \end{array} \right)$$

where, $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ be their weights and $(\partial(1), \partial(2), ..., \partial(n))$ be any rearrangement of $(1, 2, ..., n)$ with $\mathtt{C}_{\partial(j)} \le \mathtt{C}_{\partial(j-1)}$.

5 Comprehensive Evaluation Methods and Case Study Analysis

As is usually recognized, multi-attribute indicators are often interconnected, presenting a challenge in effectively integrating their information. The goal to addressing this matter lies in choosing approaches that can merge these attributes in a normal and efficient way. The T-SFDWA method stands out as one of the simplest yet most practical tools for characterizing these complex relationships. This approach offers a novel solution for integrating multi-attribute information, enhancing decision-making processes. By leveraging the T-SFDWA operator, one can more effectively navigate the complexities of multi-attribute analysis.

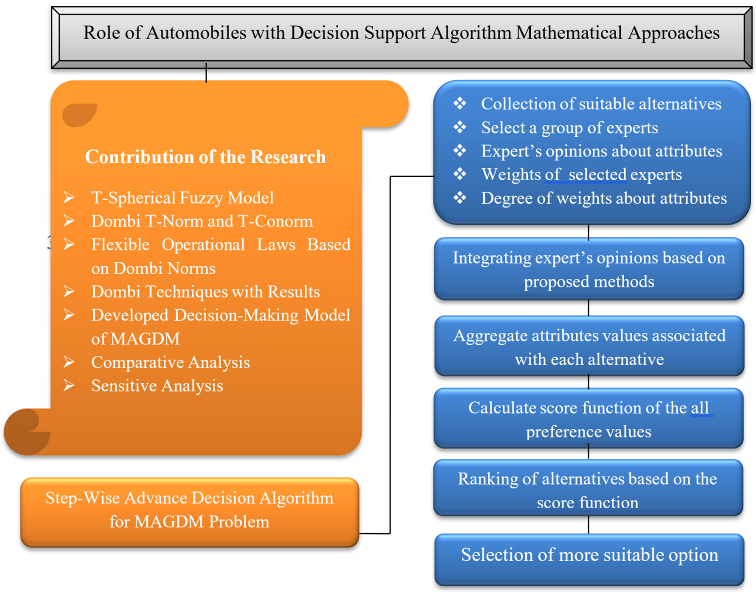

Figure 5. Flow chart of advance algorithm for MAGDM problem

Advance Algorithm: Let $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_m\}$ be a finite group of options, $C = \{C_1, C_2, ..., C_n\}$ be a finite group of key criteria. Let $E = \{E_1, E_2, ..., E_k\}$ be a finite group of experts with weights $w = (w_1, w_2, ..., w_k)$. The goal appears to be to make a decision or choice among the alternatives based on the preferences of the decision-makers and the characteristics of the options. The weighted vectors for options, alternatives, and decision-makers suggest that there are different factors or criteria being considered, each with its own importance. Figure 5 explores the stepwise algorithm of the MAGDM problem in a flow chart.

6 Illustrative Example

In the modern world, automobiles play a crucial role in making travel faster and easier. They allow people to go long distances comfortably and quickly, connecting cities and countries. With cars, buses, and other vehicles, traveling has become more accessible to everyone, improving daily life and work.

Case study: Automobile performance analysis

Automobile travel makes it easier for people to reach their destinations comfortably and quickly. Cars offer freedom and suitability, allowing individuals to travel long distances for work, leisure, or daily activities. As the automobile market continues to grow, competition among major car companies has become fierce, with each striving to earn profits. To stay ahead, car companies not only offer essential services but also focus on purchasing high-performance branded cars to improve their market position. To remain competitive, it is important for these companies to choose reliable manufacturers that provide cars with good quality, safety, and fuel efficiency. This ensures that customers are satisfied with their purchase, and the company can maintain a strong reputation. In the next section, we will explore an example to highlight the benefits of this strategy and explain how the right selection criteria can make a significant difference in the industry. The following Figure 6 explores the stepwise algorithm.

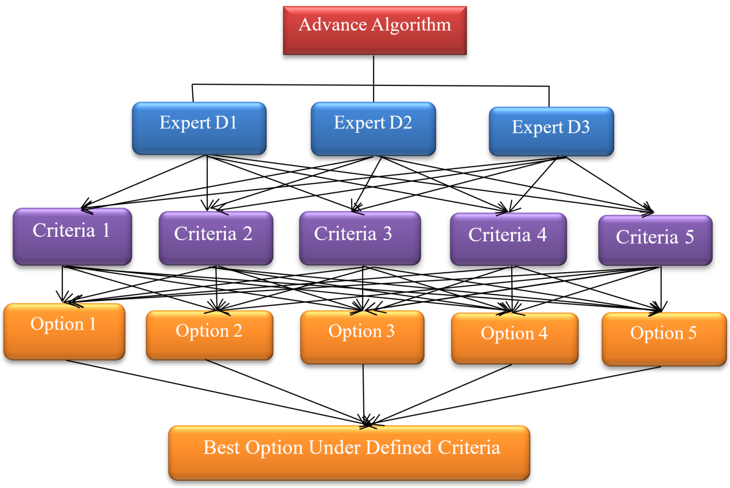

Figure 6. Flow chart of the proposed problem

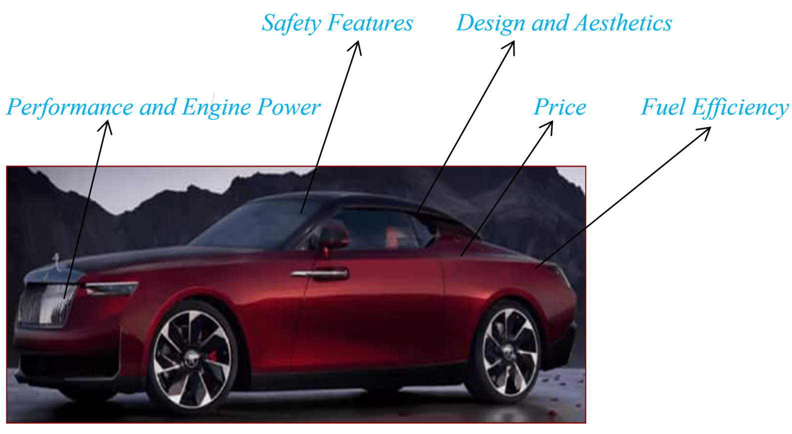

Figure 7. Simulation diagram of the four attribute indicators

The behavior of a client in choosing a car depends on many factors like price, brand, comfort, and features. People often compare different models and ask for opinions before making a decision. Their final choice is usually based on a mix of personal needs, budget, and trust in the company. The car company is evaluating five potential manufacturers (q_1, q_2, q_3, q_4, q_5) for a new batch of cars based on five key attributes: y_1: Price, y_2: Design and Aesthetics, y_3: Fuel Efficiency, y_4: Safety Features, y_5: Performance and Engine Power. To ensure an optimal decision, the company has three decision-makers (d_1, d_2, d_3) with weights $w = (0.4, 0.3, 0.3)$ to assess these manufacturers. The evaluation will be based on the experts' subjective judgments using T-SFNs, which allow for more realistic and flexible ratings compared to traditional crisp numbers. T-SFNs effectively capture the inherent imprecision and uncertainty in expert evaluations, especially when assessing subjective attributes like price, safety, appearance, and fuel efficiency. This allows for a more realistic and flexible decision-making process. The company's aim is to maximize profits by choosing the manufacturer that best balances these attributes. The aggregation of the experts' evaluations will help in identifying the most suitable manufacturer for the company's needs, ensuring a comprehensive and informed decision-making process. Hence, it is crucial for car companies to choose high-quality manufacturers in order to purchase cars with good quality, safety appearance, and fuel efficiency. For the sake of clarity, we first provide a simulation diagram Figure 7 of the five attribute indicators as follows:

Step 1: The initial evaluation matrix provided by the three experts, shown in Tables 1-3.

Table 1. Evaluation report of d_1

	q_1	q_2	q_3	q_4	q_5
y_1	(0.47, 0.51, 0.49)	(0.66, 0.45, 0.57)	(0.62, 0.53, 0.45)	(0.89, 0.48, 0.49)	(0.67, 0.53, 0.73)
y_2	(0.69, 0.58, 0.76)	(0.63, 0.72, 0.82)	(0.57, 0.59, 0.75)	(0.54, 0.58, 0.75)	(0.59, 0.47, 0.81)
y_3	(0.65, 0.53, 0.46)	(0.68, 0.59, 0.77)	(0.48, 0.57, 0.60)	(0.87, 0.48, 0.48)	(0.66, 0.63, 0.59)
y_4	(0.58, 0.59, 0.55)	(0.67, 0.59, 0.72)	(0.58, 0.56, 0.50)	(0.56, 0.48, 0.55)	(0.53, 0.58, 0.55)
y_5	(0.67, 0.53, 0.73)	(0.57, 0.59, 0.75)	(0.65, 0.53, 0.46)	(0.48, 0.53, 0.87)	(0.48, 0.57, 0.60)

Table 2. Evaluation report of d_2

	q_1	q_2	q_3	q_4	q_5
y_1	(0.82, 0.58, 0.45)	(0.48, 0.57, 0.60)	(0.69, 0.65, 0.56)	(0.57, 0.75, 0.66)	(0.45, 0.62, 0.53)
y_2	(0.54, 0.58, 0.75)	(0.59, 0.47, 0.81)	(0.68, 0.59, 0.77)	(0.56, 0.58, 0.52)	(0.57, 0.59, 0.75)
y_3	(0.87, 0.48, 0.48)	(0.53, 0.58, 0.55)	(0.63, 0.72, 0.82)	(0.67, 0.53, 0.76)	(0.48, 0.57, 0.60)
y_4	(0.56, 0.48, 0.55)	(0.66, 0.63, 0.59)	(0.67, 0.59, 0.72)	(0.84, 0.54, 0.52)	(0.50, 0.58, 0.56)
y_5	(0.87, 0.48, 0.55)	(0.67, 0.53, 0.73)	(0.77, 0.52, 0.73)	(0.65, 0.53, 0.46)	(0.47, 0.65, 0.53)

Table 3. Evaluation report of d_3

	q_1	q_2	q_3	q_4	q_5
y_1	(0.48, 0.57, 0.60)	(0.56, 0.48, 0.55)	(0.62, 0.53, 0.45)	(0.68, 0.59, 0.77)	(0.67, 0.53, 0.73)
y_2	(0.66, 0.45, 0.57)	(0.66, 0.63, 0.59)	(0.67, 0.59, 0.72)	(0.84, 0.54, 0.52)	(0.50, 0.58, 0.56)
y_3	(0.58, 0.59, 0.55)	(0.89, 0.48, 0.49)	(0.48, 0.57, 0.60)	(0.87, 0.48, 0.48)	(0.66, 0.63, 0.59)
y_4	(0.65, 0.53, 0.46)	(0.67, 0.59, 0.72)	(0.58, 0.56, 0.50)	(0.56, 0.48, 0.55)	(0.53, 0.58, 0.55)
y_5	(0.87, 0.48, 0.48)	(0.53, 0.58, 0.55)	(0.63, 0.72, 0.82)	(0.67, 0.53, 0.76)	(0.47, 0.51, 0.49)

Step 2: Using T-SFDWA and T-SFDGA, with $w = (0.4, 0.3, 0.3)$, set $t = 4$ and get Table 4 and Table 5.

Table 4. Collective assessment of all experts based on T-SFDWA operator

	q_1	q_2	q_3	q_4	q_5
y_1	(0.47, 0.51, 0.49)	(0.74, 0.65, 0.66)	(0.91, 0.82, 0.73)	(0.76, 0.54, 0.52)	(0.67, 0.53, 0.46)
y_2	(0.50, 0.58, 0.56)	(0.64, 0.57, 0.49)	(0.48, 0.57, 0.60)	(0.73, 0.56, 0.68)	(0.80, 0.64, 0.78)
y_3	(0.66, 0.63, 0.59)	(0.55, 0.77, 0.88)	(0.59, 0.47, 0.81)	(0.56, 0.55, 0.54)	(0.63, 0.74, 0.86)
y_4	(0.53, 0.58, 0.55)	(0.61, 0.77, 0.65)	(0.53, 0.58, 0.55)	(0.72, 0.65, 0.55)	(0.56, 0.83, 0.67)
y_5	(0.67, 0.53, 0.73)	(0.67, 0.53, 0.76)	(0.66, 0.63, 0.59)	(0.87, 0.48, 0.55)	(0.67, 0.53, 0.73)

Table 5. Collective assessment of all experts based on T-SFDWG operator

	q_1	q_2	q_3	q_4	q_5
y_1	(0.89, 0.68, 0.56)	(0.84, 0.54, 0.52)	(0.84, 0.75, 0.66)	(0.72, 0.65, 0.55)	(0.53, 0.58, 0.55)
y_2	(0.69, 0.54, 0.65)	(0.67, 0.53, 0.76)	(0.81, 0.57, 0.68)	(0.66, 0.63, 0.59)	(0.59, 0.47, 0.81)
y_3	(0.69, 0.58, 0.61)	(0.56, 0.58, 0.52)	(0.68, 0.59, 0.86)	(0.73, 0.56, 0.68)	(0.56, 0.55, 0.54)
y_4	(0.81, 0.74, 0.76)	(0.65, 0.53, 0.46)	(0.91, 0.82, 0.73)	(0.76, 0.54, 0.52)	(0.67, 0.53, 0.73)
y_5	(0.63, 0.72, 0.82)	(0.67, 0.53, 0.76)	(0.48, 0.57, 0.60)	(0.87, 0.48, 0.55)	(0.48, 0.57, 0.60)

Step 3: Using proposed methods, with $\kappa = (0.1, 0.2, 0.2, 0.2, 0.3)$ and get the preference values in Table 6.

Table 6. Explores aggregated outcomes by the deduced mathematical terminologies

Option	T-SFDWA	T-SFDOWA	T-SFDWG	T-SFDOWG
y_1	(0.83, 0.66, 0.64)	(0.67, 0.68, 0.65)	(0.75, 0.74, 0.73)	(0.67, 0.68, 0.70)
y_2	(0.67, 0.63, 0.64)	(0.81, 0.67, 0.74)	(0.86, 0.68, 0.78)	(0.69, 0.52, 0.72)
y_3	(0.66, 0.75, 0.61)	(0.76, 0.74, 0.65)	(0.71, 0.72, 0.56)	(0.64, 0.55, 0.69)
y_4	(0.78, 0.65, 0.64)	(0.78, 0.73, 0.69)	(0.76, 0.82, 0.73)	(0.69, 0.65, 0.75)
y_5	(0.65, 0.62, 0.56)	(0.77, 0.66, 0.68)	(0.72, 0.62, 0.74)	(0.66, 0.68, 0.75)

Step 4: Computing the score function. The computed results are depicted in Table 7. The ranking of alternatives based on score functions is also maintained in Table 7.

Table 7. Ranking of individuals based on computed score values

Methods	Scores					Ranking of Preferences
T-SFDWA	0.48	0.38	0.45	0.46	0.40	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$
T-SFDOWA	0.51	0.37	0.47	0.48	0.42	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$
T-SFDWG	0.62	0.48	0.54	0.58	0.52	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$
T-SFDOWG	0.61	0.43	0.51	0.56	0.48	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$

Step 5: Thus, the more suitable alternative is q_1. We show the results of alternatives based on score values in Figure 8 and Figure 9 to achieve more understanding and information about aggregated results.

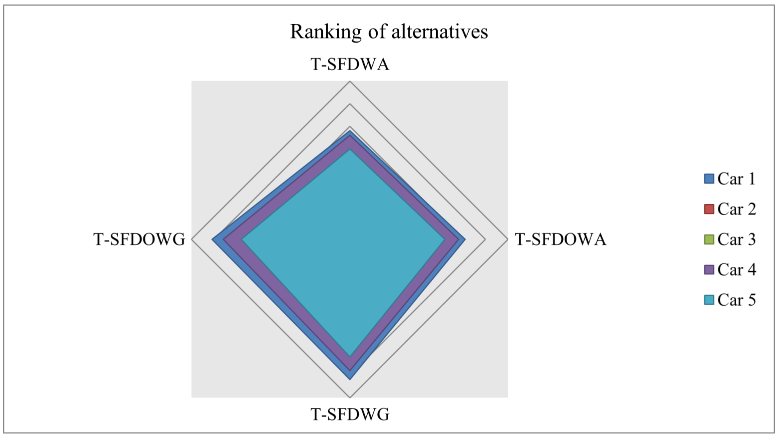

Figure 8. Ranking of all methods

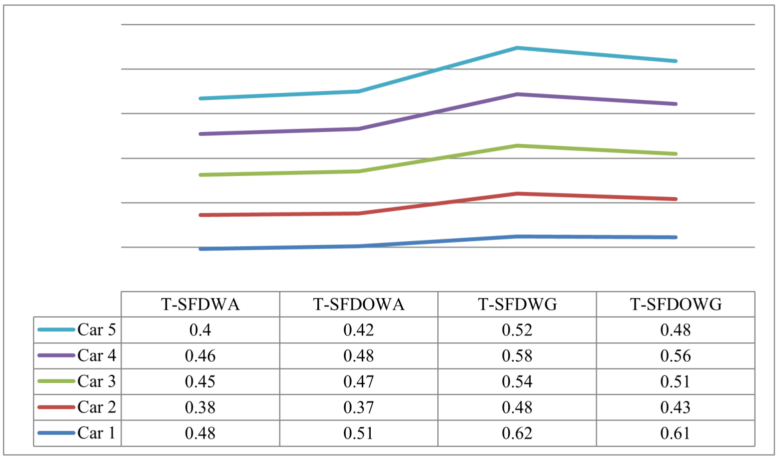

Figure 9. Graphical representation of all alternatives

7 Sensitive and Analytical Discussion

This segment explores how different decision-making models and confidence levels affect the validity and compatibility of diagnosed terminologies within the T-SF framework. By using parametric variables in operators, we aim to confirm the precision of the final outcomes and ranking of preferences. To test the reliability of these variables, we experiment with different values for the parameter "m" This helps us understand how changing "m" impacts the results of MADM. The aim is to demonstrate how adjusting this parameter can either develop or alter the decision-making process, highlighting its benefits or potential drawbacks. Table 8 demonstrates aggregated results by applying different parametric variables of based on proposed operator.

Table 8. Effects of "m" on decision making analysis

T-SFDWA	Score Function					Ranking Orders	Best Option
$m=1$	0.72	0.58	0.68	0.70	0.63	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=2$	0.70	0.54	0.66	0.67	0.60	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=3$	0.67	0.51	0.60	0.63	0.58	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=5$	0.64	0.45	0.56	0.59	0.57	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=10$	0.60	0.42	0.52	0.55	0.51	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=20$	0.57	0.39	0.49	0.51	0.43	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
T-SFDOWA	**Score Function**					**Ranking Orders**	**Best Option**
$m=1$	0.71	0.57	0.67	0.69	0.62	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=2$	0.69	0.53	0.65	0.66	0.59	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=3$	0.66	0.49	0.59	0.62	0.57	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=5$	0.63	0.44	0.55	0.58	0.56	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=10$	0.59	0.41	0.51	0.54	0.50	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=20$	0.56	0.38	0.48	0.50	0.42	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
T-SFDWG	**Score Function**					**Ranking Orders**	**Best Option**
$m=1$	0.74	0.60	0.70	0.72	0.65	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=2$	0.72	0.56	0.68	0.69	0.62	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=3$	0.69	0.53	0.62	0.65	0.60	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=5$	0.66	0.47	0.58	0.61	0.59	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=10$	0.62	0.44	0.54	0.57	0.53	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=20$	0.59	0.41	0.51	0.53	0.45	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
T-SFDOWG	**Score Function**					**Ranking Orders**	**Best Option**
$m=1$	0.68	0.54	0.64	0.66	0.59	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=2$	0.66	0.50	0.62	0.63	0.56	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=3$	0.63	0.46	0.56	0.59	0.54	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=5$	0.60	0.41	0.52	0.55	0.53	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=10$	0.56	0.38	0.48	0.51	0.47	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1
$m=20$	0.52	0.38	0.48	0.50	0.42	$q_1 \succ q_4 \succ q_3 \succ q_5 \succ q_2$	q_1

To assess the impact of operational parameters r on MADM outcomes, we will vary "m" values to rank alternatives. Results, including score functions and ranking orders for alternatives q_j $(j = 1, 2, 3, 4, 5)$ in the range of $1 \le m \le 20$, are presented in Table 8. In our study, we explored the influence of varying parametric values of m on the outcomes of our proposed aggregation operators. The results detailed in Table 8 demonstrate that increasing "m" tends to gradually minimize the outcomes of our proposed operators while maintaining the same ranking order. However, the most suitable alternative remains consistent across different parametric settings, underscoring the robustness and reliability of our methodologies.

8 Comparative Discussion

In this section, we perform a comparative analysis of our newly introduced DAO within the complex framework of T-SF environments, aiming to assess its effectiveness and applicability in decision-making scenarios. We juxtapose our proposed method against several established operators, namely IFWA, IFOWA, IFWG, IFOWG, IFEWA, IFEOWA, IFEWG, and IFEOWG. These operators are essential in decision-making, as they integrate various criteria into a single score, enhancing the accuracy and reliability of decisions. Their unique mathematical properties handle uncertainty and fuzziness effectively, improving multi-criteria decision-making processes. This leads to more precise and robust evaluations. The comparison results are summarized in Table 9.

The findings reveal that, when utilizing operators such as T-SFDWA, T-SFDOWA, T-SFDWG, T-SFDOWG, the prime choice can be determined by appropriately adjusting its parameter. Consequently, our proposed aggregation operators offer a versatile tool for decision-makers dealing with T-SF MPDA problems. In essence, our T-SF

Dombi operators not only effectively represent fuzzy information but also enhance the flexibility of the information aggregation process through parameter customization. This distinguishes our proposed operators from existing methods, as they cannot introduce flexibility into the data aggregation procedure. Thus, our innovative operators demonstrate advancements and reliability in the decision-making process involving T-SF data, offering decision-makers a more adaptable and superior approach compared to currently available methods. We show the results of alternatives based on score values in Figure 10 to achieve more understanding and information about aggregated results. Different color bars show alternatives and indicate the highest score function is appropriate.

Table 9. Comparisons of the novel methods with existing methods for $m = 2$

Methods	Scores					Best Option
IFWA [5]	0.85	0.78	0.83	0.80	0.74	q_1
IFOWA [5]	0.88	0.81	0.86	0.84	0.78	q_1
IFWG [6]	0.82	0.73	0.87	0.80	0.69	q_1
IFOWG [6]	0.84	0.79	0.90	0.81	0.77	q_1
IFEWA [3]	0.87	0.75	0.82	0.78	0.70	q_1
IFEOWA [3]	0.89	0.71	0.81	0.76	0.68	q_1
IFEWG [4]	0.89	0.77	0.91	0.87	0.71	q_1
IFEOWG [4]	0.86	0.76	0.88	0.80	0.73	q_1
T-SFDWA (proposed)	0.70	0.54	0.66	0.67	0.60	q_1
T-SFDOWA (proposed)	0.69	0.53	0.65	0.66	0.59	q_1
T-SFDWG (proposed)	0.72	0.56	0.68	0.69	0.62	q_1
T-SFDOWG (proposed)	0.66	0.50	0.62	0.63	0.56	q_1

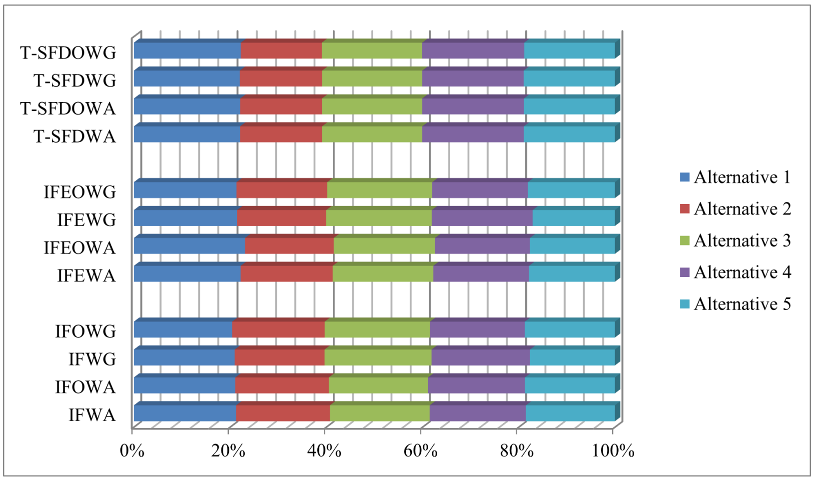

Figure 10. Comparison of the proposed methods with existing methods

9 Application of the New Methods

Below, we detail the advantages of our methods compared to others, highlighting their superior precision, flexibility, and effectiveness in handling complex decision-making scenarios.

(i) Our proposed model, such as T-SFS generalizes multiple FS models, including IFSs, PyFSs, FFSs, Q-ROFSs, PFSs and SFSs. This comprehensive framework offers enhanced flexibility and precision in representing uncertainty and ambiguity.

(ii) The existing models PFS and SFSs are special cases of our novel model. This simplification highlights the versatility and generality of our proposed operators, demonstrating their broader applicability across various FS models.

(iii) Based on the comparisons and analysis, our methods outperform existing techniques for aggregating T-SFNs. Consequently, they are better suited for addressing MADM problems, offering enhanced accuracy and effectiveness.

10 Conclusions

The T-spherical fuzzy model is a powerful mathematical tool that helps understand the relationships between multiple attribute indicators. It has been widely applied in fields such as decision-making analysis and information integration. This model and its associated operators efficiently address issues including attribute redundancy in multi-attribute indicators by balancing the influence between different attributes. In this study, we present new operators such as T-SFDWA, T-SFDOWA, and T-SFDHWA, and explore their main properties namely idempotency, boundedness, and monotonicity. The advantages of these methods are established through case studies and comparisons. However, optimizing these approaches and developing new methods for handling information with multiple interrelated attributes remains a key challenge. The goal of this study is to highlight the strengths and effectiveness of the proposed approach in solving MAGDM problems under T-spherical fuzzy information, offering a promising approach to complex decision-making scenarios.

In the future, our aim is to implement the suggested methods in various practical applications, expanding beyond their current scope. This includes applying these approaches to real-world scenarios such as medical diagnosis, pattern recognition, machine learning, and the detection of brain hemorrhages. These applications will involve adapting the techniques to operate effectively in environments that incorporate fuzzy logic extensions, enhancing their versatility and performance in diverse settings.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, 1986. https://doi.org/10.1016/S0165-0114(86)80034-3

[3] W. Z. Wang and X. W. Liu, "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations," *Int. J. Intell. Syst.*, vol. 26, no. 11, pp. 1049–1075, 2011. https://doi.org/10.1002/int.20498

[4] W. Z. Wang and X. W. Liu, "Intuitionistic fuzzy information aggregation using Einstein operations," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 5, pp. 923–938, 2012. https://doi.org/10.1109/TFUZZ.2012.2189405

[5] Z. S. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1179–1187, 2007. https://doi.org/10.1109/TFUZZ.2006.890678

[6] Z. S. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *Int. J. Gen. Syst.*, vol. 35, no. 4, pp. 417–433, 2006. https://doi.org/10.1080/03081070600574353

[7] K. Rahman, S. Abdullah, M. Jamil, and M. Y. Khan, "Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making," *Int. J. Fuzzy Syst.*, vol. 20, no. 5, pp. 1567–1575, 2018. https://doi.org/10.1007/s40815-018-0452-0

[8] K. Rahman, "Some new logarithmic aggregation operators and their application to group decision making problem based on t-norm and t-conorm," *Soft Comput.*, vol. 26, no. 6, pp. 2751–2772, 2022. https://doi.org/10.1007/s00500-022-06730-8

[9] M. M. Xia, Z. S. Xu, and N. Chen, "Induced aggregation under confidence levels," *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, vol. 19, no. 2, pp. 201–227, 2011. https://doi.org/10.1142/S0218488511006976

[10] D. J. Yu, "Intuitionistic fuzzy information aggregation under confidence levels," *Appl. Soft Comput.*, vol. 19, pp. 147–160, 2014. https://doi.org/10.1016/j.asoc.2014.02.001

[11] J. Dombi, "A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators," *Fuzzy Sets Syst.*, vol. 8, no. 2, pp. 149–163, 1982. https://doi.org/10.1016/0165-0114(82)90005-7

[12] M. R. Seikh and U. Mandal, "Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making," *Granul. Comput.*, vol. 6, pp. 473–488, 2021. https://doi.org/10.1007/s41066-019-00209-y

[13] B. C. Cường, "Picture fuzzy sets," *J. Comput. Sci. Cybern.*, vol. 30, no. 4, p. 409, 2014. https://doi.org/10.15625/1813-9663/30/4/5032

[14] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems," *J. Intell. Fuzzy Syst.*, vol. 36, no. 3, pp. 2829–2844, 2019. https://doi.org/10.3233/JIFS-172009

[15] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Comput. Appl.*, vol. 31, pp. 7041–7053, 2019. https://doi.org/10.1007/s00521-018-3521-2