Acadlore Transactions on Applied Mathematics and Statistics
https://www.acadlore.com/journals/ATAMS

Optimal Homotopy Asymptotic Treatment of the Generalised KdV-mKdV Equation for Nonlinear Dispersive Waves

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Received: 05-05-2025 **Revised:** 06-12-2025 **Accepted:** 06-22-2025

Citation: Z. Ullah, "Optimal homotopy asymptotic treatment of the generalised KdV-mKdV equation for nonlinear dispersive waves," *Acadlore Trans. Appl Math. Stat.*, vol. 3, no. 2, pp. 101–112, 2025. https://doi.org/10.56578/atams030202.

Abstract: An optimal homotopy asymptotic framework is developed for the numerical-semi-analytical treatment of the time-dependent generalized Korteweg–de Vries (KdV)-modified gKdV-mKdV equation, a prototypical nonlinear dispersive model featuring concurrent quadratic and cubic nonlinearities. The equation arises widely in optics, fluid mechanics, plasma physics and condensed-matter systems, where the accurate resolution of solitary waves and complex wave interactions is essential. The Optimal Homotopy Asymptotic Method (OHAM) is formulated without reliance on an artificial small parameter and is equipped with optimally selected convergence-control parameters, thereby overcoming limitations of classical perturbation techniques. Within this formulation, a rapidly convergent approximate analytical solution is constructed, and error dynamics are quantified against benchmark solutions. Comparative assessments indicate that OHAM attains high accuracy with modest computational effort, delivering pointwise errors and global norms that are competitive with, or superior to, those obtained by Homotopy Perturbation and Homotopy Analysis methods. The procedure is straightforward to implement, preserves the dispersive-nonlinear balance intrinsic to the gKdV–mKdV dynamics, and accommodates important special cases (KdV and mKdV limits) within a unified treatment. The approach is thus shown to provide a reliable and easily computable route to soliton-bearing solutions and other nonlinear waveforms, supporting applications in waveguides, shallow-water channels, ion-acoustic media and lattice excitations. The methodological clarity and demonstrated accuracy suggest that OHAM can serve as a practical front-line tool for nonlinear PDEs with mixed nonlinearities and higher-order dispersion, and that its convergence-control strategy can be extended to related integrable and near-integrable models.

Keywords: Optimal Homotopy Asymptotic Method (OHAM); Generalized KdV-mKdV equation; Nonlinear dispersive waves; Semi-analytical approximation; Solitons; Parameter-free homotopy; Convergence control; KdV limit; mKdV limit

1 Introduction

Over the past few decades, numerous advanced numerical techniques have been developed to yield both exact and approximate solutions for nonlinear partial differential equations (NPDEs). These methods represent a significant advancement in the fields of nonlinear sciences and theoretical physics. The effective implementation of these techniques showcases the remarkable progress made in understanding and solving NPDEs [1–9]. We focus on the Bäcklund transformation [10], the differential transformed method [11], the truncated Painlevé expansion method [12], the modified differential transformed method [13], the sine-cosine method [14], and the modified extended tanh function [15]. Beside these methods, we developed a recent method, which is known as the OHAM, to achieve the approximate analytical solutions of NLPDEs. Consider a type of NPDEs named the generalized KdV-mKdV equation [16].

$$\frac{\partial w(y,z)}{\partial z}+\left(\alpha+\beta w^{q}(y,z)+\gamma w^{2q}(y,z)\right)\frac{\partial w(y,z)}{\partial y}+\frac{\partial^{3} w(y,z)}{\partial y^{3}}=0 \quad q\geq 1 \tag{1}$$

let α, β, γ and q represent arbitrary constants. The above Eq. (1) has several special cases, which are discussed below:

1. Reduces to generalized mKdV equation, when $\alpha = \gamma \neq 0$ and $\beta = 0$.
2. Becomes to generalized KdV equation, when $\alpha \neq 0, \beta \neq 0$ and $\gamma = 0$.
3. Converts to KdV-mKdV equation, when $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$ and $q = 1$.
4. Transfer to KdV equation, when $\alpha \neq 0, \beta \neq 0, \gamma = 0$ and $q = 1$.
5. Becomes to mKdV equation, when $\alpha \neq 0, \gamma \neq 0, \beta = 0$ and $q = 1$.

This dissertation focuses on introducing the OHAM for solving NPDEs. Specifically, it addresses the higher nonlinear generalized KdV-mKdV equation. The approach leverages series solutions to construct analytical approximations for these strongly nonlinear PDEs. The aim is to fully exploit the advantages of OHAM in deriving effective approximate solutions. The OHAM is a highly effective and straightforward technique for higher-order NPDEs. This method builds on the Homotopy Analysis Method (HAM) [17], which was further advanced with the Homotopy Perturbation Method (HP Method) [18] introduced in 1998. In 2008, Vasile Marinca and colleagues enhanced these perturbation methods by developing OHAM [19], which surpasses both HAM and the HP Method in effectiveness. The OHAM [19, 20] adopts the global structure of both the HA Method and the HP Method. When $H(p) = -p$, it exactly mirrors the HP Method, and when $H(p) = ph$, it corresponds precisely to the HA Method.

In brief, the HA Method and the HP Method are special cases of OHAM.

2 The Algorithm of OHAM

Consider the boundary value problem,

$$D(w(y,z)) + f(y,z) = 0, \quad y \in \delta, z \geq 0 \tag{2}$$

$$\psi\left(w_0(y,z), \frac{\partial w_0(y,z)}{\partial z}\right) = 0, \quad z \in \Gamma \tag{3}$$

where, D represents a differential operator, ψ is a boundary operator, Γ is the boundary of the domain δ and $f(y,z)$ is a known analytic function.

Step 1: Let L and N denote the differential operators corresponding to the simple and the complicated parts, respectively, such that $D = L + N$.

Step 2: Solve the following equations for problems Eqs. (2) and (3):

$$\begin{aligned}
&L\left(w_0(y,z)\right) + f = 0, \psi\left(w_0(y,z), \frac{\partial w_0(y,z)}{\partial z}\right) = 0,\\
&L\left(w_1\left(y,z;\mathrm{C}_1\right)\right) = C_1 N_0\left(w_0(y,z)\right), \psi\left(w_1\left(y,z;\mathrm{C}_1\right), \frac{\partial w_1\left(y,z;C_1\right)}{\partial z}\right) = 0,\\
&L\left(w_r(y,z)\right) = L\left(w_{r-1}(y,z)\right) + C_r N_0\left(w_0(y,z)\right) +\\
&\sum_{j=1}^{r-1} C_j\left[L\left(w_{r-j}(y,z)\right) + N_{r-j}\left(w_0(y,z), w_1(y,z), \ldots, w_{r-j}(y,z)\right)\right],\\
&r = 2,3,\ldots, \psi\left(w_r(y,z), \frac{\partial w_r(y,z)}{\partial z}\right) = 0
\end{aligned}$$

Step 3: A modal series solution, which is to be used for the approximate solution of Eq. (2) is: $w_r\left(y,z;\mathrm{C}_i\right) = w_0(y,z) + w_1\left(y,z;C_1\right) + w_2\left(y,z;C_1,C_2\right) + \ldots .w_{r-1}\left(y,z,C_1,\ldots .\mathrm{C}_{r-1}\right) + w_r\left(y,z;C_1,\ldots .C_r\right)$.

The procedure to determine C_i is carried out as follows:

Step 4: If value of $R = \mathrm{D}(\tilde{w}(y,z)) + f(y,z)$ is zero, stop.

Step 5: If $R \neq 0$, then choose $J\left(C_1, C_2, \ldots\right) = \int_0^z \int_\delta R^2\left(y,z;C_1,C_2,C_3\ldots\right) dydz$ and evaluate $C_i, i = 1,2,3,\ldots\ldots r$ using $\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} = \ldots\ldots.. = 0$.

3 Application of OHAM to the Problem GKdV-mKdV

In this section, the significance and originality of the OHAM for finding analytical approximate solutions to NLPDEs have been demonstrated. Several cases of the KdV–mKdV equations were solved to assess the efficiency of the OHAM. The results obtained by OHAM exhibited excellent agreement with the corresponding exact solutions, thereby confirming both the accuracy and reliability of the proposed approach.

Problem 1: Consider the generalized KdV equation

$$\frac{\partial w(y,z)}{\partial z} + \left(\alpha + \beta w^q(y,z)\right)\frac{\partial w(y,z)}{\partial y} + \frac{\partial^3 w(y,z)}{\partial y^3} = 0 \tag{4}$$

The solitary wave solution is:

$$w(\varepsilon)=\left(\frac{A}{2\beta}\left(q^2+3q+2\right)\operatorname{Sec}h^2\left(\frac{q\sqrt{A}\varepsilon}{2}\right)\right)^{\frac{1}{q}}$$

where, $\varepsilon=y-(\alpha+\beta)z,\beta>0$.

Condition:

$$w(0,z)=\left(\frac{A}{2\beta}\left(q^2+3q+2\right)\operatorname{Sec}h^2\left(\frac{q\sqrt{A}(\alpha+\beta)z}{2}\right)\right)^{\frac{1}{q}}$$

$$w(y,0)=\left(\frac{A}{2\beta}\left(q^2+3q+2\right)\operatorname{Sec}h^2\left(\frac{q\sqrt{A}y}{2}\right)\right)^{\frac{1}{q}}$$

where, A,α,β and q are arbitrary constants. As regard the OHAM's basic selection in the start of the procedure, we will choose $L(F(y,z;q))$ and $\mathrm{N}(F(y,z;q))$ for Eq. (4), so that:

$$L(F(y,z;q))=\frac{\partial F(y,z;q)}{\partial z}$$

$$\mathrm{N}(F(y,z;q))=(\alpha+\beta(F(y,z;q))^q)\frac{\partial F(y,z;q)}{\partial y}+\frac{\partial^3 F(y,z;q)}{\partial y^3}$$

with boundary conditions:

$$\mathrm{F}(y,0;q)=\left(\frac{A\left(q^2+3q+2\right)}{2\beta}\operatorname{Sec}h^2\left(\frac{1}{2}q\sqrt{A}y\right)\right)^{\frac{1}{q}}$$

Using the method of OHAM, the zeroth-order linear partial differential (LPD) equation as:

$$\frac{\partial w(y,z)}{\partial z}=0,\ \text{with } w_0(y,0)=\left(\frac{A}{2\beta}\left(q^2+3q+2\right)\operatorname{Sec}h^2\left(\frac{q\sqrt{A}y}{2}\right)\right)^{\frac{1}{q}}$$

Solution of the zeroth-order problem is $w_0(y,z)=\left(\frac{A(q+2)(q+1)}{8\beta\cosh\left(\mathrm{A}^{1/2}qz\right)^2}\right)^{1/q}$.

The first-order problem:

$$\frac{\partial w_1(y,z;\mathrm{C})}{\partial z}=C_1N_0\left(w_0(y,z)\right)=\mathrm{C}_1\left((\alpha+\beta u_0^q(y,z))\frac{\partial w_0(y,z)}{\partial y}+\frac{\partial^3 w_0(y,z)}{\partial y^3}\right),w_1(y,0)=0$$

Solution of the first-order problem is:

$$w_1(y,z)=\left(2^{(q-3)/q}A^{(q+2)/2q}C_1 s\sinh\left(A^{1/2}qy\right)\left(q^2+3q+2\right)^{1/q}\right.$$
$$\left(\frac{15A}{4}+\frac{45Aq^2}{8}-a\cosh\left(A^{1/2}qy\right)^2-4A\cosh\left(\mathrm{A}^{1/2}qy\right)+\frac{15Aq^2}{8}\right)/\beta^{1/q}\cosh\left(A^{1/2}qy\right)^{(3q+2)/q}$$

The second-order problem:

$$\frac{\partial w_2\left(y,z;C_1,C_2\right)}{\partial z}=C_2N_0\left(w_0(y,z)\right)+C_1N_1\left(w_0(y,z),w_1\left(y,z;C_1\right)\right)+\left(1+C_1\right)L\left(w_1\left(y,z;C_1\right)\right)$$
$$=C_2\left((\alpha+w_0^q(y,z)\beta)\frac{\partial w_0(y,z)}{\partial y}+\frac{\partial^3 w_0(y,z)}{\partial y^3}\right)+C_1\left(\left(\alpha+\beta w_1^q\left(y,z;C_1\right)\right)\frac{\partial w_1\left(y,z;C_1\right)}{\partial y}+\frac{\partial^3 w_1(y,z)}{\partial y^3}\right)$$
$$+\left(1+C_1\right)\frac{\partial w_1\left(y,z;C_1\right)}{\partial y}$$

with boundary solution $w_2(y,0)=0$ and the solution will be $w_2,w_2\left(y,z;C_1,C_2\right)=w_2$.

Using w_0,w_1 and w_2 to obtain the 2nd order approximation of Eq. (4) as:

$$\tilde{w}\left(y,z;C_1,C_2\right)=w_0(y,z)+w_1\left(y,z;C_1\right)+w_2\left(y,z;C_1,C_2\right)\tag{5}$$

Residual of Eq. (5) is:

$$R(y,z;C_1,C_2) = \frac{\partial}{\partial z}\tilde{w}(y,z) + \left(\alpha + \beta\tilde{w}^q(y,z)\right)\frac{\partial}{\partial y}\tilde{w}(y,z) + \frac{\partial^3}{\partial y^3}\tilde{w}(y,z)$$

Values of the constants are:

$$C_1 = 0.03877042104787104556056288075 43$$
$$C_2 = -0.02112048559587924298491861731 40$$

Table 1. OHAM's solution, exact solution and error of Eq. (4), when $\alpha = 0.006, \beta = 0.05, A = 0.0004$, and $q = 1$

z	y	w_{oham}	Exact	$L_\infty * 10^8$
	1	0.00599759977784	0.00599763124533	3.1467486437
1	2	0.00599040850653	0.00599047144215	6.2935619859
	3	0.00597844914763	0.00597854335653	9.4208900491
	1	0.00599760020887	0.00599761596713	1.5758262646
0.5	2	0.00599040936865	0.00599044086086	3.1492212110
	3	0.00597845044105	0.00597849756971	4.7128657727
	1	0.00599760055366	0.00599760370924	0.3155575614
0.1	2	0.005990410058311	0.00599041636066	0.6302346723
	3	0.00597845147577	0.00597846090537	0.9429604671
	1	0.00599760059676	0.00599760217479	0.1578032999
0.05	2	0.00599041014452	0.00599041329594	0.3151417379
	3	0.00597845160511	0.00597845632015	0.4715044404
	1	0.00599760063124	0.00599760094688	0.0315645830
0.01	2	0.00599041021348	0.00599041084381	0.0630322519
	3	0.00597845170858	0.00597845265162	0.0943047612

Table 2. OHAM's solution, exact solution and error of 2nd order approximations of Eq. (4), when $A = 0.002$, $\alpha = 0.0005, \beta = 5, q = 2$

z	y	w_{oham}	Exact	$L_\infty * 10^6$
	1	0.02439785356	0.02439797199	0.118439690
	2	0.02410930036	0.02410956749	0.26714413
1.5	3	0.02364039101	0.02364085971	0.46869257
	4	0.02300861045	0.02300934358	0.73312810
	5	0.02223624727	0.02223730347	1.05619228
	6	0.02134860351	0.02135002492	1.42140679
	1	0.02439760975	0.02439768088	0.07112709
	2	0.02410883541	0.02410899581	0.16040031
0.9	3	0.02363974790	0.02364002929	0.28139461
	4	0.02300784459	0.02300828472	0.44011544
	5	0.02223541829	0.02223605229	0.63399101
	6	0.02134776757	0.02134862069	0.85312599
	1	0.02439728412	0.02439729204	0.00791238
	2	0.02410821509	0.02410823293	0.01783912
0.1	3	0.02363889022	0.02363892151	0.03129259
	4	0.02300682347	0.02300687240	0.04893705
	5	0.02223431316	0.02223438365	0.07048427
	6	0.02134665325	0.02134674808	0.09483353
	1	0.02439726375	0.02439726771	0.00395648
	2	0.02410817630	0.02410818522	0.00892009
0.05	3	0.02363883661	0.02363885225	0.01564712
	4	0.02300675965	0.02300678412	0.02446963
	5	0.02223424409	0.02223427934	0.03524341
	6	0.02134658362	0.02134663103	0.04741807

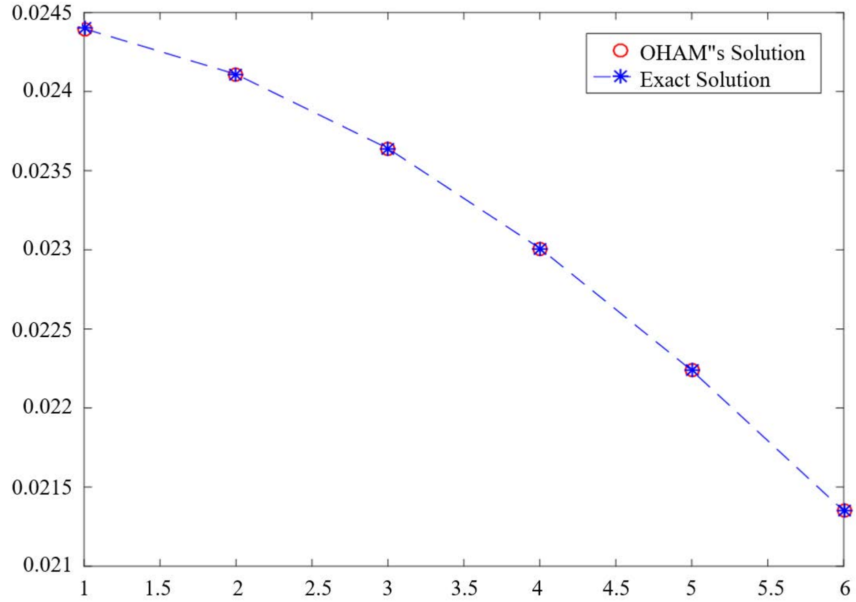

Figure 1. Approximate solution and exact solution of problem 1 at $A = 0.002, \alpha = 0.0005, \beta = 5, q = 2$

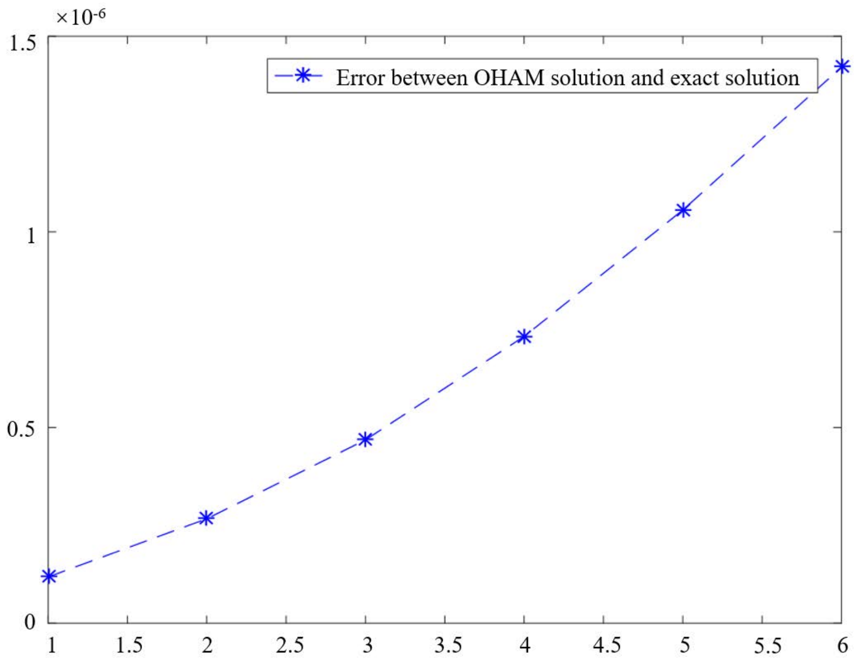

Figure 2. Error between approximate solution and exact solution of problem 1

Using the above-mentioned values of C_1 and C_2 into Eq. (5), we obtained the approximate solution of Eq. (4). The 2nd order approximate solutions obtained by OHAM, together with the exact solutions and the corresponding errors, are presented in Table 1, Table 2, Figure 1 and Figure 2, where the errors were quantified by $L_\propto = |w_{ex}(y,z) - w_{OHAM}(y,z)|$. Figure 1 shows that the OHAM's solution closely agrees with the exact solution, which is indicates the high accuracy and effectiveness of the OHAM method. Figure 2 illustrates the error between the exact and approximate solutions. The error is minimal, further supporting the method's precision. These results confirm the reliability of OHAM in providing accurate approximations.

Problem 2: Consider the 2nd special case generalized mKdV equation

$$\frac{\partial w(y,z)}{\partial z} + \left(\alpha + \gamma w^{2q}\right) \frac{\partial w(y,z)}{\partial y} + \frac{\partial^3 w(y,z)}{\partial y^3} = 0 \tag{6}$$

The solitary wave solution is:

$$w(y,z)=\left(\sqrt{\frac{A}{\gamma}\left(2q^2+3q+1\right)}\,\mathrm{Sech}(q\sqrt{A}\varepsilon)\right)^{\frac{1}{q}},A>0,\gamma>0,\varepsilon=y-(\alpha+A)z$$

where, conditions are:

$$w(0,z)=\left(\sqrt{\frac{A}{\gamma}\left(2q^2+3q+1\right)}\,\mathrm{Sec}\,h(q\sqrt{A}(\alpha+\beta)z)\right)^{\frac{1}{q}}$$
$$w(y,0)=\left(\sqrt{\frac{A}{\gamma}\left(2q^2+3q+1\right)}\,\mathrm{Sec}\,h(q\sqrt{A}z)\right)^{\frac{1}{q}}$$

As in the study [19], again choose L and N of Eq. (6) as:

$$L(F(y,z;q))=\frac{\partial F(y,z;q)}{\partial z}$$
$$N(F(y,z;q))=\left(\alpha+\gamma F^{2q}(y,z;q)\right)\frac{\partial F(y,z;q)}{\partial y}+\frac{\partial^3 F(y,z;q)}{\partial y^3}$$

with condition:

$$F(y,0)=\left(\sqrt{\frac{A}{\gamma}\left(2q^2+3q+1\right)}\,\mathrm{Sec}\,h(q\sqrt{A}y)\right)^{\frac{1}{q}}$$

Again, construct a sequence of problems, the zeroth-order problem of this sequence is:

$$\frac{\partial w_0(y,z)}{\partial z}=0,\ \text{with } w_0(y,0)=\left(\sqrt{\frac{A}{\gamma}\left(2q^2+3q+1\right)}\,\mathrm{Sec}\,h(q\sqrt{A}y)\right)^{\frac{1}{q}}$$

Table 3. OHAM's solution, exact solution and error of problem 2 by using $\alpha=0.001,\gamma=1,A=0.0003$

z	y	w_{oham}	Exact	$L_\infty * 10^6$
1.5	-1	0.25892423349	0.25892402988	0.2036080
	1	0.25892443250	0.25892463552	0.2030180
	3	0.25830519282	0.25830579907	0.6062510
	5	0.25707681068	0.25707781013	0.9994540
	7	0.25525940149	0.25526077795	1.3764570
	9	0.25288198284	0.25288371466	1.7318150
1	-1	0.25892426666	0.25892413098	0.1356730
	1	0.25892439933	0.25892453474	0.1354110
	3	0.25830509387	0.25830549810	0.4042310
	5	0.25707664758	0.25707731394	0.6663650
	7	0.25525917689	0.25526009459	0.9176970
	9	0.25288170027	0.25288285487	1.1545980
0.5	-1	0.25892429982	0.25892423202	0.0678040
	1	0.25892436616	0.25892443390	0.0677380
	3	0.25830499492	0.25830519706	0.2021480
	5	0.25707648448	0.25707681769	0.3332140
	7	0.25525895229	0.25525941116	0.4588780
	9	0.25288141769	0.25288199502	0.5773260
0.1	-1	0.25892432636	0.25892431280	0.0135560
	1	0.25892433963	0.25892435318	0.0135530
	3	0.25830491575	0.25830495619	0.0404350
	5	0.25707635399	0.25707642064	0.0666480
	7	0.25525877260	0.25525886438	0.0917800
	9	0.25288119163	0.25288130710	0.1154690

Table 4. The absolute error of OHAM approximation with exact and approximate solution of problem 2 for $A = 0.0003, \alpha = 0.001, \gamma = 2, q = 2$

z	y	w_{oham}	Exact	$L_\infty * 10^7$
1.5	-3	0.217207411429	0.217206901147	5.1028146827
	-1	0.217728459761	0.217728288548	1.7121288077
	1	0.217728627112	0.217728797829	1.7071702653
	3	0.217207910685	0.217208420479	5.0979387960
1	-3	0.217207494638	0.217207154505	3.4013346973
	-1	0.217728487653	0.217728373566	1.1408682585
	1	0.217728599220	0.217728713086	1.1386644619
	3	0.217207827476	0.217208167393	3.3991676366
0.5	-3	0.217207577848	0.217207407808	1.7003964683
	-1	0.217728515544	0.217728458529	0.57015865540
	1	0.2177285713282	0.217728628289	0.569607706
	3	0.2172077442666	0.217207914252	1.699854703
0.1	-3	0.2172076444152	0.217207610412	0.340035953
	-1	0.2177285378579	0.217728526459	0.113987655
	1	0.2177285490147	0.217728560411	0.113965617
	3	0.2172076776989	0.217207711700	0.340014282

Table 5. OHAM's result, exact result, and error of Eq. (6) by using $\alpha = 0.0005, \gamma = 5, A = 0.002, q = 1$

z	y	w_{oham}	Exact	$L_\infty * 10^7$
0.9	-2	0.0487943428329	0.0487940485292	2.94303686
	0	0.0489897948557	0.0489897946077	0.00248011
	2	0.0487946306778	0.0487949244953	2.93817504
0.5	-2	0.0487944067984	0.0487942433564	1.63442027
	0	0.0489897948557	0.0489897947791	0.00076547
	2	0.0487945667122	0.0487947300042	1.63291971
0.1	-2	0.0487944707639	0.0487944380875	0.32676401
	0	0.0489897948557	0.0489897948526	0.00003062
	2	0.0487945027467	0.0487945354171	0.32670399
0.05	-2	0.0487944787596	0.0487944624222	0.16337450
	0	0.0489897948557	0.0489897948549	0.00000766
	2	0.0487944947509	0.0487945110869	0.16335949
0.01	-2	0.0487944851562	0.0487944818888	0.03267369
	0	0.0489897948557	0.0489897948556	0.00000031
	2	0.0487944883545	0.0487944916218	0.03267309

Solution of the zeroth-order problem is:

$$w_0(y,z) = \left(\left(\frac{A}{\gamma}\left(2q^2+3q+1\right)\right)^{1/2} \operatorname{sech}\left(A^{1/2}qy\right)\right)^{\frac{1}{q}}$$

The first-order problem is:

$$\frac{\partial w_1(y,z;C_1)}{\partial z} = C_1 N_0\left(w_0(y,z)\right) = C_1\left(\left(\alpha + \gamma w_0^{2q}(y,z)\right)\frac{\partial w_0(y,z)}{\partial y} + \frac{\partial^3 w_0(y,z)}{\partial y^3}\right), \text{ and } w_1(y,0) = 0$$

Solution of the first-order problem is:

$$w_1(y,z;C_1) = -\left(\mathrm{A}^{(q+1)/2q} C_1 z (A+\alpha)\left(2q^2+3q+1\right)^{1/2q} \sinh\left(\mathrm{A}^{1/2}qy\right)\right) / \left(\gamma^{1/2q}\cosh\left(A^{1/2}qy\right)^{(q+1)/q}\right)$$

Using $w_0(y,z)$ and $w_1(y,z;C_1)$, we can get OHAM's solution of Eq. (6):

$$\tilde{w}(y,z;C_1) = w_0(y,z) + w_1(y,z;C_1) \tag{7}$$

Residual of Eq. (6) is given by:

$$R(y,z;C_1) = \frac{\partial}{\partial z}\tilde{w}(y,z) + \left(\alpha + \gamma \tilde{w}^{2q}(y,z)\right)\frac{\partial}{\partial y}\tilde{w}(y,z) + \frac{\partial^3}{\partial y^3}\tilde{w}(y,z)$$

The constant value C_1 can be obtained using the above-mentioned method in Section 2 , that is:

$$C_1 = -0.3286027860054143227619684904 8$$

By introducing constant C_1 into Eq. (7), we obtain the approximate solution of problem 2. The results are also given in Tables 3–5, Figure 3 and Figure 4 of the same problem. Figure 3 shows that exact solution and approximate solution are closed to each other, while Figure 4 shows the absolute error of OHAM's solution and exact solution.

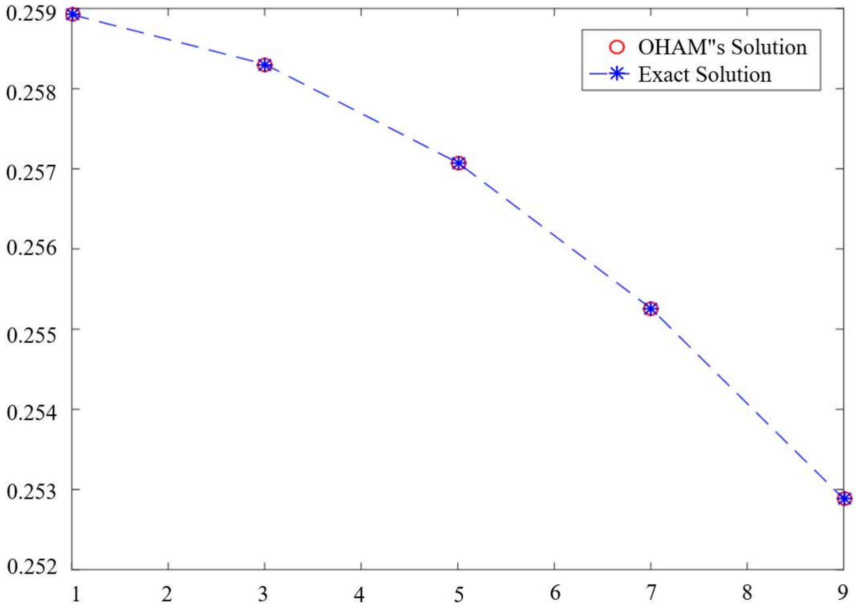

Figure 3. OHAM's solution and exact solution of problem 2 at $\alpha = 0.001, \gamma = 1, A = 0.0003$

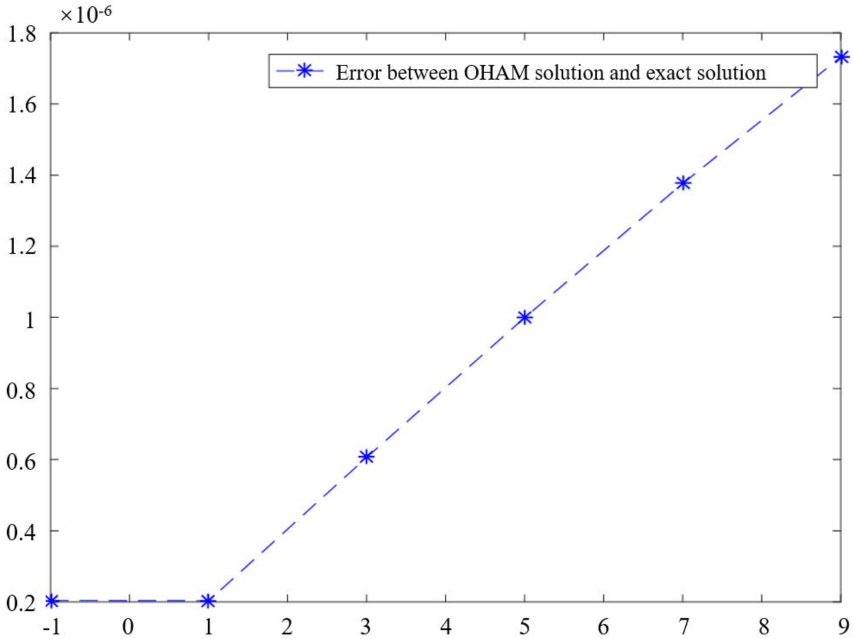

Figure 4. Error between exact and approximate solution of problem 2

Problem 3: Consider the Generalized KdV-mKdV

$$\frac{\partial w(y,z)}{\partial z} + \left(\alpha + \beta w^q(y,z) + \gamma w^{2q}(y,z)\right)\frac{\partial w(y,z)}{\partial y} + \frac{\partial^3 w(y,z)}{\partial y^3} = 0, \quad q > 0 \tag{8}$$

with the exact solution: $w(y,z)\beta < 0, \gamma > 0, \varepsilon = y - \alpha z$, where, the conditions are: $w(0,z)$ and $w(y,0)$. By taking L and N for problem 3 as:

$$L(F(y,z;q)) = \frac{\partial F(y,z;q)}{\partial z}$$
$$N(F(y,z;q)) = \left(\alpha + \beta F^q(y,z;q) + \gamma F^{2q}(y,z;q)\right)\frac{\partial F(y,z;q)}{\partial y} + \frac{\partial^3 F(y,z;q)}{\partial y^3}$$

with conditions: $F(0,z;q)$ and $F(y,0;q)$.

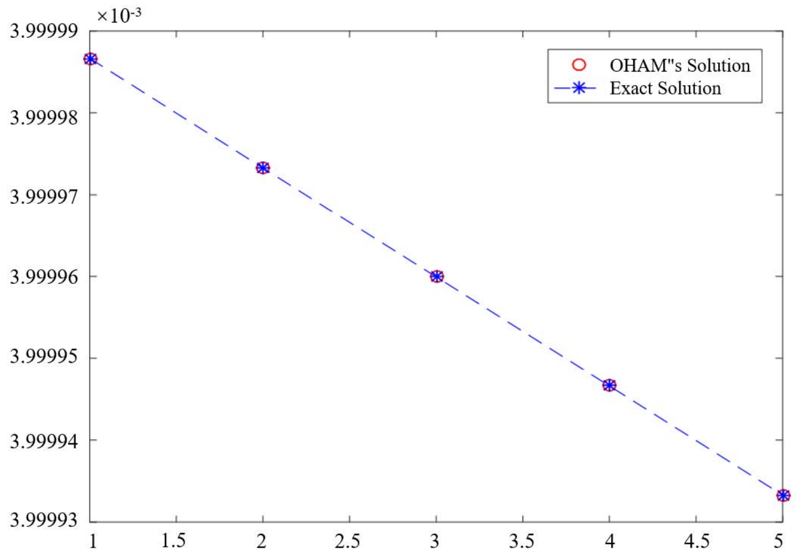

Figure 5. OHAM's solution and exact solution of problem 3 at $\alpha = 0.05, \beta = 0.01, \gamma = 5, q = 1$

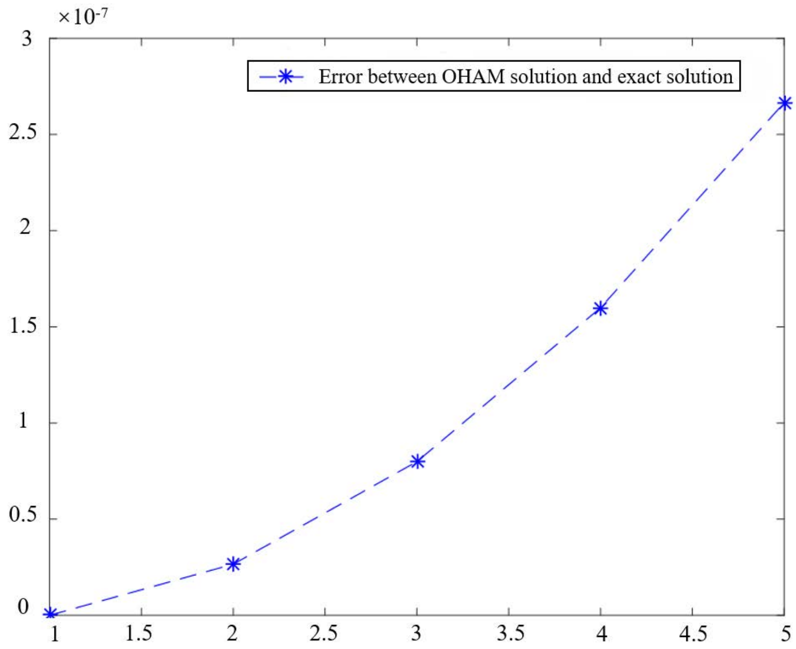

Figure 6. Error between exact and approximate solution of problem 3

Table 6. OHAM's result, exact result and errors of Eq. (8) taking $\alpha = 0.05, \beta = 0.01, \gamma = 5, q = 1$

z	y	w_{oham}	**Exact**	L_∞
0.001	1	0.00399998661661	0.00399998666804	$5.14322592 * 10^{-11}$
	2	0.00399997328341	0.00399994667004	$2.6613368146 * 10^{-8}$
	3	0.00399995995030	0.00399988000759	$7.994270208 * 10^{-8}$
	4	0.00399994662	0.00399978668338	$1.59933903235 * 10^{-7}$
	5	0.00399993328	0.00399966670111	$2.66583239143 * 10^{-7}$
0.05	1	0.00399998416	0.00399998673329	$2.5715312615 * 10^{-9}$
	2	0.00399997085	0.00399994680062	$2.40279552391 * 10^{-8}$
	3	0.003999957495	0.00399988020351	$7.72919804898 * 10^{-8}$
	4	0.003999944162	0.00399978694459	$1.57217880799 * 10^{-7}$
	5	0.003999930829	0.00399966702764	$2.63801926309 * 10^{-7}$
0.01	1	0.003999986166	0.00399998668004	$5.143195908 * 10^{-10}$
	2	0.003999972833	0.00399994669404	$2.61384843835 * 10^{-8}$
	3	0.003999959499	0.00399988004359	$7.94558228448 * 10^{-8}$
	4	0.0039999461664	0.00399978673137	$1.594350299701 * 10^{-7}$
	5	0.0039999328335	0.00399966676109	$2.660723737667 * 10^{-7}$
0.1	1	0.0039999816568	0.00399998679971	$5.1428957924 * 10^{-9}$
	2	0.00399996832365	0.00399994693370	$2.13899438571 * 10^{-8}$
	3	0.00399995499057	0.00399988040324	$7.45873275795 * 10^{-8}$
	4	0.00399994165758	0.00399978721099	$1.54446594350 * 10^{-7}$
	5	0.00399992832468	0.00399966736067	$2.609640169732 * 10^{-7}$

Table 7. Distinct results of Eq. (9) at $\alpha = 0.0005, \beta = 0.0007, \gamma = 2, q = 3$

z	y	w_{oham}	**Exact**	L_∞
0.05	0	0.09932883882831	0.09932883883793	$9.61717358263 * 10^{-12}$
	1	0.09932883371784	0.09932883372771	$9.87269497939 * 10^{-12}$
	2	0.09932882860737	0.09932881839657	$1.0210802989959 * 10^{-8}$
	3	0.09932882349691	0.09932879284452	$3.06523909503605 * 10^{-8}$
	4	0.09932881838644	0.09932875707158	$6.131485963463961 * 10^{-8}$
	5	0.09932881327598	0.09932871107781	$1.021981648706080 * 10^{-7}$
0.09	0	0.09932883882062	0.09932883883793	$1.731090784925 * 10^{-11}$
	1	0.09932883371015	0.09932883372792	$1.777084636343 * 10^{-11}$
	2	0.09932882859968	0.09932881839698	$1.02027004217097 * 10^{-8}$
	3	0.09932882348921	0.09932879284513	$3.064408396575131 * 10^{-8}$
	4	0.09932881837875	0.09932875707240	$6.13063482344274 * 10^{-8}$
	5	0.09932881326829	0.09932871107884	$1.0218944905580 * 10^{-7}$
0.5	0	0.09932883874176	0.09932883883793	$9.617144835881 * 10^{-11}$
	1	0.09932883363129	0.09932883373001	$9.872666232664 * 10^{-11}$
	2	0.0993288285208	0.09932881840117	$1.01196493328797 * 10^{-8}$
	3	0.0993288234104	0.09932879285142	$3.055893760922868 * 10^{-8}$
	4	0.09932881829989	0.09932875708078	$6.121910661797351 * 10^{-8}$
	5	0.099328813189425	0.09932871108931	$1.021001121897659 * 10^{-7}$
1	0	0.09932883864558	0.09932883883793	$1.9234225790098 * 10^{-10}$
	1	0.09932883353512	0.09932883373257	$1.9745268583705 * 10^{-10}$
	2	0.09932882842465	0.09932881840628	$1.001836809855508 * 10^{-8}$
	3	0.09932882331418	0.09932879285908	$3.045510117039903 * 10^{-8}$
	4	0.09932881820372	0.09932875709100	$6.111271498410321 * 10^{-8}$
	5	0.09932881309325	0.09932871110209	$1.019911653734744 * 10^{-7}$

Similarly, the 2nd order approximation of Eq. (8) is given as:

$$\tilde{w}\left(y, z; C_1, C_2\right) = w_0(y, z) + w_1\left(y, z; C_1\right) + w_2\left(y, z; C_1, C_2\right) \tag{9}$$

Again, for residual $R\left(y, z; C_1, C_2\right)$, we obtain constants C_1 and C_2 via the same method in Section 2 as:

$$C_1 = -10.2942528120446051834007004954\,5$$

$$C_2 = -11.7129319993073606540606299\,27$$

By substituting C_1 and C_2 into Eq. (9), and we get the OHAM's solution of problem 3. Moreover, the efficiency of problem 3 is shown in Table 6, Table 7, Figure 5, and Figure 6, where we justify that OHAM's technique is a powerful method for the solution of linear and non-linear PDEs.

4 Conclusions

In this research work, we have systematically investigated the applicability and proficiency of the Optimal Homotopy Asymptotic Method (OHAM) in solving a class of non-linear partial differential equations (PDEs), with a particular focus on the generalized KdV-mKdV equation and its special cases. Our study demonstrates that OHAM is not only a flexible and robust analytical technique but also a highly accurate one for treatment of complex nonlinear systems. Unlike many traditional numerical or perturbation-based methods, OHAM does not rely on small or large parameters, making it more broadly applicable and less restrictive.

The achievement of OHAM lies in its ability to systematically construct a convergent series solution through an optimal selection of auxiliary functions and parameters. This approach allows for rapid convergence to the exact or highly accurate approximate solutions with relatively a few iterations. The OHAM's implementation is straightforward and does not require discretization, linearization, or perturbation, which are common limitations in many other analytical or semi-analytical techniques.

All necessary calculations and simulations for the generalized KdV-mKdV equation and its various special cases were performed using MATLAB. The computational results confirm the method's capability to handle nonlinearities effectively and produce solutions that are consistent with those available in this research work.

Overall, this work confirms that OHAM is a powerful tool for researchers working with nonlinear PDEs. Its ease of implementation, high accuracy, and wide applicability make it a valuable addition to the existing methods for solving complex mathematical models arising in physics, engineering, and applied sciences.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The author declares no conflict of interest.

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