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Parametric Similarity Measurement of T-Spherical Fuzzy Sets for Enhanced Decision-Making

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Abstract: The T-spherical fuzzy set (T-SFS), an advancement over the spherical fuzzy set (SFS), offers a refined approach for addressing contradictions and ambiguities in data. In this context, similarity measures (SMs) serve as critical tools for quantifying the resemblance between fuzzy values, traditionally relying on the calculation of distances between these values. Nevertheless, existing methodologies often encounter irrational outcomes due to certain characteristics and complex operations involved. To surmount these challenges, a novel parametric similarity measure is proposed, grounded in three adjustable parameters. This enables decision-makers to tailor the SM to suit diverse decision-making styles, thereby circumventing the aforementioned irrationalities. An analytical comparison with existing SM reveals the superiority of the proposed measure through mathematical validation. Furthermore, the utility of this measure is demonstrated in the resolution of multi-attribute decision-making (MADM) problems, highlighting its efficacy over several existing approaches within the domain of T-SFS. The implementation of the proposed SM not only enhances the precision of similarity assessment in fuzzy sets but also significantly contributes to the optimization of decision-making processes.

Keywords: Fuzzy set (FS); Spherical fuzzy set (SFS); T-spherical fuzzy set (T-SFS); Similarity measure (SM); Multi-attribute decision-making (MADM)

1 Introduction

The process of extracting and analyzing information from real-world problems is fraught with ambiguity and uncertainty. There have been numerous initiatives to lessen that uncertainty. There have been numerous initiatives to lessen that uncertainty. Zadeh popularized the idea of the FS [1], which is a well-known method of reducing uncertainty. The notion of the extension of the crisp set in practice characterizes an object's belongingness through the membership degree (MD). Atanassov [2] tried to reduce the uncertainty by coming up with the idea of intuitionistic FS (IFS). This expanded on the FS idea and defined an object's belongingness by both MD and non-membership degree (NMD). By using the MD and NMD as intervals from [0, 1], Atanassov and Gargov [3] formalized the interval-valued IFS in order to obtain greater accuracy during the information extraction process. Numerous researchers in a variety of domains, including pattern recognition [4], decision-making [5], and medical diagnosis [6], among others, used IFS. The MD and NMD could only have a limited amount of space assigned to them by IFS since the sum of the MD and NMD did not fall inside. The IFS was therefore a narrowly scoped framework. The IFS was made bigger by Yager [7] and Yager and Ronald [8], who came up with the ideas of the Pythagorean FS (PyFS) and the q-rung orthopair FS (qROFS).

Because IFS, PyFS, and qROFS can reduce vagueness in information extraction, their applications have a lot of potential in real-world scenarios. However, because these tools only have two degrees for an element's description, there are some scenarios in which they are unable to extract information without information loss. Cuong [9] presented the picture FS (PFS) with an extra degree called the abstinence degree (AD) to illustrate the belongingness of an object using three degrees. Numerous academics have used the PFS, as seen in [10]. However, there were times when the PFS concept failed because it was not met. For instance, the MD, AD, and NMD have respective values of. In this instance. As such, the PFS was an extremely constrained method. In order to broaden the scope of the PFS, Mahmood et al. [11] presented the notions of T-SFS and SFS. Therefore, the most recent framework that can extract information with the highest level of accuracy is the T-SFS.

Finding the best option from a list of options can be done with interest using the MADM technique. The fuzzy theory's introduction has completely transformed and advanced MADM. Many academics have used various environments to improve the MADM process. Khan et al. [12] employed complicated T-SFS to address the MADM issue. Interval-valued IFS was utilized by Senapati et al. [13] to address the MADM problem. PyFS was utilized by Jana et al. [14] to address the MADM issue. PFS was utilized by Senapati and Tapan [15] to address the MADM issue. Khan et al. [16] Employed sophisticated T-SFS to address the MADM issue. Mahmood and Ali [17] solved the MADM problem by using complex single-valued neutrosophic (CSVNS). Employed sophisticated T-SFS to address the MADM issue. Riaz and Farid [18] employed a complex PFS to address the MADM issue. Khan et al. [19] employed sophisticated T-SFS to address the MADM issue. Riaz et al. [19] employed bipolar FS to address the MADM issue. Garg [20] IFS to address the MADM issue. Interval-valued PFS was utilized by Ashraf et al. [21] to address the MADM issue. Garg [22] employed PyFS to address the MADM issue. Pamucar et al. [23] employed qROFS to address the MADM issue. Sarfraz et al. [24] gave the idea of prioritized aggregation operators. A. Hussain and Pamucar [25] develop the theory of MADM using the rough fuzzy set. Hussain et al. [26] give the concept of MADM on the basis of the Aczel-Alsina aggregation operator. Sarfraz [27] developed the theory of MADM using the application of T-SFS. Ullah et al. [28] develop the theory of Aczel-Alsina using the application of MADM.

When assessing the degree of similarity between two fuzzy values (FVs), SM is an important tool. Numerous academics have been introduced to intriguing applications in pattern recognition, MADM, and medical diagnosis based on various SM types. The SMs for the IFS framework are introduced by Boran and Akay [4] and Du and Hu [29], where it is also discussed how to apply the developed SMs to pattern recognition. The SM for qROFS was presented by Donyatalab et al. [30], and the SMs for PyFS were presented in [31], along with a discussion of their intriguing applications. The base of the cosine function and the contingent function, respectively, for the PFS are the cosine SMs and contingent SMs, which Wei [32] introduced and applied to the MADM. Dice SM was created by Wei and Geo [30] for the PFS. A few SMs for PFS were presented by Dinh and Thao [33] and applied to the MADM problem. By taking into account the PFS's refusal degree, Singh et al. [34] expanded the SMs and used them to address the clustering issue. The SMs were introduced by Luo and Zhang [35] and are based on a few fundamental PFS operations. The idea of the SM for SFS was presented by Rafiq et al. [36] and applied to the MADM. Zhao et al. [37] created the SMs for the T-SFS framework and used them in both MADM and the pattern recognition problem. presented the SMs for the T-SFS and used them in the pattern recognition process Shen et al. [38]. Ullah et al. [39] presented the SMs for the T-SFS framework and used them to solve the pattern recognition problem. In a similar vein,Shen [38] described the SMs for the T-SFS and their uses in pattern recognition. Jin et al. [40] presented the SMs for the T-SFS and then used them to solve the pattern recognition and medical diagnosis problems. Below are some key takeaways from the SMs that were previously discussed.

• Due to the limited amount of information that these frameworks are able to extract from real-world scenarios, all of the SMs for the IFS, PYFS, qROFS, and PFS are outdated. Because of the uncertainty and information loss, decision-makers are unable to arrive at the best decisions. As a result, the T-SFS should define the advanced SMs that can more accurately determine how similar two FVs are.

• In some unique circumstances, a few of the SMs covered above fail to compute. For instance, because of the division by zero problems, certain SMs do not provide the decision results.

• As a result, the main contribution of this study is to enhance the SMs' capacity for identification and address their shortcomings; in order to do this, a new SM must be proposed.

First, some fundamental ideas are covered in Section 2 of this paper. Section 3 presents an overview of the current standard methods of SMs and addresses their limitations. The new SM for T-SFS is developed and uses the parameters to improve and generalize the current SMs for T-SFS. The application of the suggested SMs to the MADM problem is shown in Section 4, and the study is summarized in Section 5.

2 Preliminaries

This section presents some basic concept to understand the article.

Definition 1 [2]: The shape of an IFS on a set X is $X = \{(\mu, (\alpha, \varphi')) : 0 \leq \alpha u\hbar\,(\alpha(\mu), \varphi'(\mu)) \leq 1\}$. Moreover, the pair (α, φ') is referred to as an intuitionist FV (IFV), and $r(\mu) = 1 - \alpha u\hbar\,(\alpha, \varphi')$ denotes the hesitancy degree of $\mu \in X$.

Definition 2 [7]: A PyFS with the form $X = \left\{(\mu, (\alpha, \varphi')) : 0 \leq \alpha u\hbar\left(\alpha^2(\mu), \varphi'^2(\mu)\right) \leq 1\right\}$ exists on a set X. Additionally, the hesitancy degree of $\mu \in X$ is represented by $r(\mu) = 1 - \alpha u\hbar\left(\alpha^2(\mu), \varphi'^2(\mu)\right)$, and the pair (α, φ') is referred to as a Pythagorean FV (PyFV).

Definition 3 [9]: A PFS of the form $X = \{(\mu, (\alpha, \tau, \varphi')) : 0 \leq \alpha u\hbar\,(\alpha(\mu), \tau(\mu), \varphi'(\mu)) \leq 1\}$ exists on a set X. Additionally, the refusal degree of $\mu \in X$ is represented by $r(\mu) = 1 - \alpha u\hbar\,(\alpha(\mu), \tau(\mu), \varphi'(\mu))$ and the pair (α, τ, φ') is referred to as a picture FV(PFV).

Definition 4 [11]: A SFS for every universal set X has the form $X = \{(\mu, (\alpha, \tau, \varphi')) : \forall \mu \in X\}$. Here, α, τ, and φ' are mappings form $X \rightarrow [0,1]$ denoting MD, AD, and NMD respectively, provided that $0 \leq \operatorname{sum}\left(\alpha^2(\mu), \tau^2(\mu), \varphi'^2(\mu)\right) \leq 1$ and $r(\mu) = \sqrt{1 - \operatorname{sum}\left(\alpha^2(\mu), \tau^2(\mu), \varphi'^2(\mu)\right)}$ are known as the refusal degree (RD) of $r(\mu)$. A spherical FV (SFV) is thought to be the triplet (α, τ, φ').

Definition 5 [11]: A T-SFS has the form $X = \{(\mu, (\alpha, \tau, \varphi')) : \forall \mu \in X\}$ for any universal set X. Here, α, τ, and φ' are mappings form $X \rightarrow [0,1]$ denoting MD, AD, and NMD, respectively, provided that for some $\eta \in z^+$ $0 \leq \operatorname{sum}\left(\alpha^\eta(\mu), \tau^\eta(\mu), \varphi'^\eta(\mu)\right) \leq 1$ and $r(\mu) = \sqrt{1 - \operatorname{sum}\left(\alpha^2(\mu), \tau^2(\mu), \varphi'^2(\mu)\right)}$ is known as the RD of $r(\mu)$. It is believed that the triplet (α, τ, φ') is a T-spherical FV (T-SFV).

Remark 1:

Let $\eta = 2$ so T-SFS degenerate SFS.

Let $\eta = 1$ so T-SFS degenerate PFS.

Let $\eta = 2, \tau = 0$ so T-SFS degenerate PyFS.

Let $\eta = 1, \tau = 0$ T-SFS degenerate IFS.

Let $\eta = 1, \tau = \varphi = 0$ so TSFS degenerate FS.

Definition 6 [11]: Let $\varsigma = \left\{\left(\mu, \alpha_\varsigma^\eta(\mu), \tau_\varsigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu)\right) \mid \mu \in X\right\}$ and $\sigma = \{(\mu, \alpha_\sigma^\eta(\mu), \tau_\sigma^\eta(\mu), \varphi_\sigma'^\eta(\mu)) \mid \mu \in X\}$, then

$\varsigma \subseteq \sigma$ If and only if $\alpha_\varsigma^\eta(\mu) \leq \alpha_\sigma^\eta(\mu), \tau_\varsigma^\eta(\mu) \geq \tau_\sigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu) \geq \varphi_\sigma'^\eta(\mu)$. For $\mu \in X$.

$\varsigma = \sigma$ If and only if $\varsigma \subseteq \sigma$ and $\sigma \subseteq \varsigma$

i.e., $\alpha_\varsigma^\eta(\mu) = \alpha_\sigma^\eta(\mu), \tau_\varsigma^\eta(\mu) = \tau_\sigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu) = \varphi_\varsigma'^\eta(\mu)$,

$\varsigma^c = \left\{\left(\mu, \alpha_\varsigma^\eta(\mu), \tau_\varsigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu)\right) \mid \mu \in X\right\}$.

Definition 7 [37]: Let $\varsigma = \left\{\left(\mu, \alpha_\varsigma^\eta(\mu), \tau_\varsigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu)\right) \mid \mu \in X\right\}$ and $\sigma = \{(\mu, \alpha_\sigma^\eta(\mu), \tau_\sigma^\eta(\mu), \varphi_\sigma'^\eta(\mu)) \mid \mu \in X\}$ be any two be T-SFSs on universe X, then the SM among ς and σ is demarcated as $\beta(\varsigma, \sigma)$, which fulfills the following axioms:

(ϑ_1) $0 \leq \beta(\varsigma, \sigma) \leq 1$;

(ϑ_2) $\beta(\varsigma, \sigma) = 1$ Iff $\varsigma = \sigma$;

(ϑ_3) $\beta(\varsigma, \sigma) = \beta(\sigma, \varsigma)$;

(ϑ_4) Let C be any T-SFS such that $\varsigma \subseteq \sigma \subseteq C$, then $\beta(\varsigma, C) \leq \beta(\varsigma, \sigma)$ and $\beta(\varsigma, C) \leq \beta(\sigma, C)$;

Now, we evaluate some standing similarity measures between T-SFSs in the behind.

Let $\varsigma = \left\{\left(\mu_\tau, \alpha_\varsigma(\mu_\tau), \tau_\varsigma(\mu_\tau), \varphi'_\varsigma(\mu_\tau)\right) \mid \mu_\tau \in X\right\}$ and $\sigma = \{(\mu_\tau, \alpha_\sigma(\mu_\tau), \tau_\sigma(\mu_\tau), \varphi'_\sigma(\mu_\tau)) \mid \mu_\tau \in X\}$ be any two be T-SFSs on $X = \{\mu_1, \mu_2, \ldots, \mu_\eta\}, \rho_\varsigma(\mu_\tau)$ and $\rho_\sigma(\mu_\tau)$ be the refusal degrees of element μ_τ belonging to T-SFSs ς and σ respectively, where $\rho_\varsigma(\mu_\tau) = 1 - \alpha_\varsigma^\eta(\mu_\tau), \tau_\varsigma^\eta(\mu_\tau), \varphi_\varsigma'^\eta(\mu_\tau)$ and $\rho_\sigma(\mu_\tau) = 1 - \alpha_\sigma^\eta(\mu_\tau), \tau_\sigma^\eta(\mu_\tau), \varphi_\sigma'^\eta(\mu_\tau)$. The existing similarity degrees between T-SFSs ς and σ are reviewed as follows:

where, $\tau = 1, 2, 3 \ldots \eta$.

The SMs defined by Shen et al. [38] based on the T-SFS is given as follows.

$$\beta_1(\varsigma, \sigma) = 1 - \frac{1}{2\eta} \sum_{\tau=1}^{\eta} \left(\left|\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right| + \left|\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right| + \left|\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right| + \left|\, \rho_\varsigma^\eta(\mu_\tau) - \rho_\sigma^\eta(\mu_\tau)\, \right|\right) \tag{1}$$

$$\beta_2(\varsigma, \sigma) = 1 - \frac{1}{2\eta} \sum_{\tau=1}^{\eta} \left|\left(\left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\varphi'_\varsigma(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right)\right)\right| \tag{2}$$

$$\beta_3(\varsigma, \sigma) = \frac{1}{4\eta} \sum_{\tau=1}^{\eta} \begin{pmatrix} \left|\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right| + \left|\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right| \\ + \left|\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right| + \left|\rho_\varsigma^\eta(\mu_\tau) - \rho_\sigma^\eta(\mu_\tau)\right| \end{pmatrix} + \sum_{\tau=1}^{\eta} \begin{pmatrix} \left|\, \left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \right. \\ \left. \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right) \,\right| \end{pmatrix} \tag{3}$$

$$\beta_4(\varsigma, \sigma) = 1 - \frac{1}{\eta} \sum_{\tau=1}^{\eta} \left(\left|\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau) \vee \right| \tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau) \,|\, \vee \left|\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\, \right|\right) \tag{4}$$

$$\beta_5(\varsigma,\sigma)=1-\frac{1}{\eta}\sum_{\tau=1}^{\eta}\frac{1-\left(\left|\alpha_{\varsigma}^{\eta}(\mu_\tau)-\alpha_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\tau_{\varsigma}^{\eta}(\mu_\tau)-\tau_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\varphi_{\varsigma}'(\mu_\tau)-\varphi_{\sigma}'(\mu_\tau)\right|\right)}{1+\left(\left|\alpha_{\varsigma}^{\eta}(\mu_\tau)-\alpha_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\tau_{\varsigma}^{\eta}(\mu_\tau)-\tau_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\varphi_{\varsigma}'^{\eta}(\mu_\tau)-\varphi_{\sigma}'(\mu_\tau)\right|\right)} \tag{5}$$

$$\beta_6(\varsigma,\sigma)=\frac{\sum_{\tau=1}^{\eta}1-\left(\left|\alpha_{\varsigma}^{\eta}(\mu_\tau)-\alpha_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\tau_{\varsigma}^{\eta}(\mu_\tau)-\tau_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\varphi_{\varsigma}'^{\eta}(\mu_\tau)-\varphi_{\sigma}'^{\eta}(\mu_\tau)\right|\right)}{\sum_{\tau=1}^{\eta}1+\left(\left|\alpha_{\varsigma}^{\eta}(\mu_\tau)-\alpha_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\tau_{\varsigma}^{\eta}(\mu_\tau)-\tau_{\sigma}^{\eta}(\mu_\tau)\right|\vee\left|\varphi_{\varsigma}'^{\eta}(\mu_\tau)-\varphi_{\sigma}'^{\eta}(\mu_\tau)\right|\right)} \tag{6}$$

$$\beta_7(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau)\right)+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\wedge\tau_{\sigma}^{\eta}(\mu_\tau)\right)+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\wedge\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)}{(\alpha_{\tau}^{\eta})\vee\alpha_{\sigma}^{\eta}(\mu_\tau))+(\tau_{\varsigma}^{\eta}(\mu_\tau)\vee\tau_{\sigma}^{\eta}(\mu_\tau))+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\vee\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)} \tag{7}$$

$$\beta_8(\varsigma,\sigma)=\frac{\sum_{\tau=1}^{\eta}\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau)\right)+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\wedge\tau_{\sigma}^{\eta}(\mu_\tau)\right)+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\wedge\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)}{\sum_{\tau=1}^{\eta}(\alpha_{\varsigma}^{\eta}(\mu_\tau)\vee\alpha_{\sigma}^{\eta}(\mu_\tau))+(\tau_{\varsigma}^{\eta}(\mu_\tau)\vee\tau_{\sigma}^{\eta}(\mu_\tau))+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\vee\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)} \tag{8}$$

$$\beta_9(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau)\right)+\left(1-\tau_{\varsigma}^{\eta}(\mu_\tau)\right)\wedge(1-\tau_{\sigma}^{\eta}(\mu_\tau))+\left(1-\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)\wedge(1-\varphi_{\sigma}'^{\eta}(\mu_\tau))}{(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau))+(1-\tau_{\varsigma}^{\eta}(\mu_\tau))\vee(1-\tau_{\sigma}^{\eta}(\mu_\tau))+\left(1-\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)\vee\left(1-\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)} \tag{9}$$

$$\beta_{10}(\varsigma,\sigma)==\frac{\sum_{\tau=1}^{\eta}\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau)\right)+\left(1-\tau_{\varsigma}^{\eta}(\mu_\tau)\right)\wedge(1-\tau_{\sigma}^{\eta}(\mu_\tau))+\left(1-\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)\wedge(1-\varphi_{\sigma}'^{\eta}(\mu_\tau))}{\sum_{\tau=1}^{\eta}(\alpha_{\varsigma}^{\eta}(\mu_\tau)\wedge\alpha_{\sigma}^{\eta}(\mu_\tau))+(1-\tau_{\varsigma}^{\eta}(\mu_\tau))\vee(1-\tau_{\sigma}^{\eta}(\mu_\tau))+(1-\varphi'^{\eta}(\mu_\tau))\vee\left(1-\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)} \tag{10}$$

Ullah et al. [41] provided the following SMs for the T-SFS environment, which are based on the cosine function.

$$\beta_{11}(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\cdot\alpha_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\cdot\tau_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\cdot\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}{\sqrt{(\alpha_{\varsigma}^{\eta}(\mu_\tau))^2+(\tau_{\varsigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)^2}\sqrt{(\alpha_{\sigma}^{\eta}(\mu_\tau))^2+(\tau_{\sigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}} \tag{11}$$

$$\beta_{12}(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\omega_\tau\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\cdot\alpha_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\cdot\tau_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\cdot\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}{\sqrt{(\alpha_{\varsigma}^{\eta}(\mu_\tau))^2+(\tau_{\varsigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)^2}\sqrt{(\alpha_{\sigma}^{\eta}(\mu_\tau))^2+(\tau_{\sigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}} \tag{12}$$

$$\beta_{13}(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\cdot\alpha_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\cdot\tau_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\cdot\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}{\left((\alpha_{\varsigma}^{\eta}(\mu_\tau))^2+(\tau_{\varsigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)^2\right)\cdot\left((\alpha_{\sigma}^{\eta}(\mu_\tau))^2+(\tau_{\sigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2\right)} \tag{13}$$

$$\beta_{14}(\varsigma,\sigma)=\frac{1}{\eta}\sum_{\tau=1}^{\eta}\omega_\tau\frac{\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\cdot\alpha_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\tau_{\varsigma}^{\eta}(\mu_\tau)\cdot\tau_{\sigma}^{\eta}(\mu_\tau)\right)^2+\left(\varphi_{\varsigma}'^{\eta}(\mu_\tau)\cdot\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2}{\left((\alpha_{\varsigma}^{\eta}(\mu_\tau))^2+(\tau_{\varsigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\varsigma}'(\mu_\tau)\right)^2\right)\cdot\left((\alpha_{\sigma}^{\eta}(\mu_\tau))^2+(\tau_{\sigma}^{\eta}(\mu_\tau))^2+\left(\varphi_{\sigma}'^{\eta}(\mu_\tau)\right)^2\right)} \tag{14}$$

3 A Review of A Few T-spherical Fuzzy Similarity Metrics that are Currently in Use

Using the SM as an arithmetic tool to determine the degree of similarity between substances, decision-making, clinical determination, and example acknowledgment problems have all been resolved. Despite the fact that many SMs between T-SFSs have been proposed, their practical application can lead to irrational and counterintuitive outcomes that pose significant challenges for functional clients. In this section, we thoroughly examine a few of the SMs that are currently in use from a mathematical perspective, as shown in Table 1 in the subsequent.

One of the fundamental axioms of T-spherical SMs is ϑ_2. It is evident from examining Table 1 that the similarity measures $\beta_5, \beta_7, \beta_8, \beta_{12}$, and β_{14} do not meet this axiom. The following is a detailed discussion:

(1) Let $\varsigma=\left\{\left(\mu_\tau,\alpha_{\varsigma}^{\eta}(\mu_\tau),\tau_{\varsigma}^{\eta}(\mu_\tau),\varphi_{\varsigma}'^{\eta}(\mu_\tau)\right)\mid\mu_\tau\in X\right\}$ and $\sigma=\{(\mu_\tau,\alpha_{\sigma}^{\eta}(\mu_\tau),\tau_{\sigma}^{\eta}(\mu_\tau),\varphi_{\sigma}'^{\eta}(\mu_\tau))\mid\mu_\tau\in X\}$ be any two be T-SFSs on $X=\{\mu_1,\mu_2,\ldots,\mu_\eta\}$. As demonstrated below, there are two situations in which the similarity measure β_{12} does not meet the axiom $(\vartheta_2)\ \beta(\varsigma,\sigma)=1$ implies $\varsigma=\sigma$.

Table 1. A thorough examination of a few of the current similarity metrics for T-SFS

β	Doesn't Adhere to the Principle β_2	By Zero Issue Significant Loss of Information	Significant Loss of Information
β_1	Yes	NO	NO
β_2	Yes	NO	NO
β_3	Yes	NO	NO
β_4	Yes	NO	NO
β_5	NO	Yes	NO
β_6	Yes	NO	NO
β_7	Yes	NO	NO
β_8	NO	NO	Yes
β_9	Yes	NO	NO
β_{10}	Yes	NO	NO
β_{11}	Yes	NO	NO
β_{12}	NO	NO	NO
β_{13}	Yes	NO	NO
β_{14}	NO	NO	NO
β_m	Yes	NO	NO

If $\alpha_{\varsigma}^{\eta}(\mu_\tau) = \tau_{\varsigma}^{\eta}(\mu_\tau) = \varphi_{\varsigma}^{\prime\eta}(\mu_\tau) \neq \alpha_{\sigma}^{\eta}(\mu_\tau) = \tau_{\sigma}^{\eta}(\mu_\tau) = \varphi_{\sigma}^{\prime\eta}(\mu_\tau)$
i.e. $\varsigma \neq \sigma$ based on Eq. (12), we have

$$\beta_{12}(\varsigma,\sigma) = \frac{1}{\eta}\sum_{\tau=1}^{\eta}\omega_\tau \frac{\alpha_{\varsigma}^{\eta}(\mu_\tau)\,\alpha_{\sigma}^{\eta}(\mu_\tau) + \tau_{\varsigma}^{\eta}(\mu_\tau)\,\tau_{\sigma}^{\eta}(\mu_\tau) + \varphi_{\varsigma}^{\prime\eta}(\mu_\tau)\,\varphi_{\sigma}^{\prime\eta}(\mu_\tau)}{\sqrt{(\alpha_{\varsigma}^{\eta}(\mu_\tau))^2 + (\tau_{\varsigma}^{\eta}(\mu_\tau))^2 + (\varphi_{\varsigma}^{\prime\eta}(\mu_\tau))^2}\sqrt{(\alpha_{\sigma}^{\eta}(\mu_\tau))^2 + (\tau_{\sigma}^{\eta}(\mu_\tau))^2 + (\varphi_{\sigma}^{\prime\eta}(\mu_\tau))^2}}$$
$$= \omega_\tau \frac{3\alpha_{\varsigma}^{\eta}(\mu_\tau)\,\alpha_{\sigma}^{\eta}(\mu_\tau)}{\sqrt{3\alpha_{\varsigma}^{\eta}(\mu_\tau)}\sqrt{3\alpha_{\sigma}^{\eta}(\mu_\tau)}} = 1$$

If $\alpha_{\varsigma}^{\eta}(\mu_\tau) = 2\alpha_{\sigma}^{\eta}(\mu_\tau), \tau_{\varsigma}^{\eta}(\mu_\tau) = 2\tau_{\sigma}^{\eta}(\mu_\tau) \quad a\eta\varphi \quad \varphi_{\varsigma}^{\prime\eta}(\mu_\tau) = 2\varphi_{\sigma}^{\prime\eta}(\mu_\tau)$
i.e., $\varsigma \neq \sigma$ based on Eq. (12), we have

$$\beta_{12}(\varsigma,\sigma) = \frac{1}{\eta}\sum_{\tau=1}^{\eta}\omega_\tau \frac{\alpha_{\varsigma}^{\eta}(\mu_\tau)\,\alpha_{\sigma}^{\eta}(\mu_\tau) + \tau_{\varsigma}^{\eta}(\mu_\tau)\,\tau_{\sigma}^{\eta}(\mu_\tau) + \varphi_{\varsigma}^{\prime\eta}(\mu_\tau)\,\varphi_{\sigma}^{\prime\eta}(\mu_\tau)}{\sqrt{(\alpha_{\varsigma}^{\eta}(\mu_\tau))^2 + (\tau_{\varsigma}^{\eta}(\mu_\tau))^2 + (\varphi_{\varsigma}^{\prime\eta}(\mu_\tau))^2}\sqrt{(\alpha_{\sigma}^{\eta}(\mu_\tau))^2 + (\tau_{\sigma}^{\eta}(\mu_\tau))^2 + (\varphi_{\sigma}^{\prime\eta}(\mu_\tau))^2}}$$
$$= \omega_\tau \frac{2\left(\alpha_{\varsigma}^{\eta}(\mu_\tau)\right)^2 + 2\left(\tau_{\sigma}^{\eta}(\mu_\tau)\right)^2 + 2\left(\varphi_{\sigma}^{\prime\eta}(\mu_\tau)\right)^2}{\sqrt{4(\alpha_{\varsigma}^{\eta}(\mu_\tau))^2 + 4(\tau_{\varsigma}^{\eta}(\mu_\tau))^2 + 4(\varphi_{\varsigma}^{\prime\eta}(\mu_\tau))^2}\sqrt{(\alpha_{\sigma}^{\eta}(\mu_\tau))^2 + (\tau_{\sigma}^{\eta}(\mu_\tau))^2 + (\varphi_{\sigma}^{\prime\eta}(\mu_\tau))^2}} = 1$$

Obviously, the SM β_{12} is invalid in the aforementioned cases.

(2) The similar SMs $\beta_5, \beta_7, \beta_8, \beta_{12}$, and β_{14} do not satisfy the the axiom $\beta(\varsigma,\sigma) = 1$ implies $\varsigma = \sigma$ and in this instance, these SMs give practical users a result that defies logic.

(3) For the SMs $\beta_3, \beta_7, \beta_8, \beta_{11}, \beta_{12}, \beta_{13}$ and β_{14}, when TSFSs $A = \sigma = (\mu, 0.0, 0.0, 0.0)$ defined on $X = \{\mu\}$, they do not $\beta_3, \beta_7, \beta_8, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}$ satisfy this axiom $(\vartheta_2)\, \beta(\varsigma,\sigma) = 1$ Iff $\varsigma = \sigma$; and definition 7 is not fill full these operators. These SMs are invalid in this instance because they don't adhere to the axiom.

(4) The articulation structure and the data it contains do not fully settle the SM's ability to detect the proximity of fuzzy. The identification ability becomes more grounded the more data the SM concentrates on. Upon examining Table 1, we discover that the SMδ_1 solely takes into account the variations in positive, neutral, negative, or refusal degrees between T-SFSs, resulting in a significant loss of information. As an illustration, let $\varsigma = (0.1, 0.2, 0.1), \sigma = (0.6, 0.2, 0.1)$ be two T-SFSs. Since $|0.1 - 0.1| < |0.2 - 0.2| <| 1 - 0.1 - 0.2 - 0.1| < |1 - 0.6 - 0.2 - 0.1| < |0.1 - 0.6\ |$, hence, the SM among A and σ only uses the SM β_1 to take into account the difference in the positive degree between ς and σ. In this case, the SM will result in significant data loss in a workable application, making it unable to give more precise results to users in real-world scenarios. Furthermore, we discover that in this instance, the SMs $\beta_2, \beta_4, \beta_5, \beta_6, \beta_{p_3}$ have the same disadvantage.

3.1 A Measure of Parametric Similarity Between Fuzzy T-spherical Sets

In order to overcome the limitations of the current SMs, we expand a parametric SM between T-SFSs in this section, taking into consideration the explanations for the absurd results of the above analysis in Table 1.

In this section, by developing a paired capability, we provide a parametric T-spherical fuzzy SM. The investigation in Table 1 designates that the SMs $\beta_5, \beta_7, \beta_8, , \beta_{12}$, and β_{14} have the shortcomings. As a result, the parametric SMs are described in Definition 8.

Definition 8: Let $\varsigma = \{(\mu_\tau, \alpha_\varsigma(\mu_\tau), \tau_\varsigma(\mu_\tau), \varphi'{}_\varsigma(\mu_\tau)) \mid \mu_\tau \in X\}$ and $\sigma = \{((\mu_\tau, \alpha_\sigma(\mu_\tau), \tau_\sigma(\mu_\tau), \varphi'_\sigma(\mu_\tau))) \mid \mu_\tau \in X\}$ be any two be T-SFSs on $X = \{\mu_1, \mu_2, \ldots, \mu_\eta\}$, then the function $\beta_\hbar : T-\mathrm{SFS}(\mu) \times T-\mathrm{SFS}(\mu) \to [0,1]$ is defined by:

$$\beta_\hbar(\varsigma,\sigma) = 1 - \left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta} \Delta_{1\varsigma\sigma}^p(\mu_\tau) + \Delta_{2\varsigma\sigma}^p(\mu_\tau) + \Delta_3^p\varsigma\sigma(\mu_\tau)\right]^{\frac{1}{p}} \tag{15}$$

$\beta_h(\varsigma,\sigma)$ is a similarity measure between ς and σ, where,

$\Delta_{1\varsigma\sigma}(\mu_\tau) = \dfrac{1}{\hbar_1+1}\left|\hbar_1\left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right)\right| \hbar_1 \in [0,+\infty)$,

$\Delta_{2\varsigma\sigma}(\mu_\tau) = \dfrac{1}{2\hbar_2+1}\left|\hbar_2\left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right)\right| \hbar_2 \in [0,+\infty)$,

$\Delta_{3\varsigma\sigma}(\mu_\tau) = \dfrac{1}{2\hbar_3+1}\left|\hbar_3\left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right) - \left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right)\right| \hbar_3 \in [0,+\infty)$,

$\dfrac{1}{\hbar_1+1} + \dfrac{1}{2\hbar_2+1} + \dfrac{1}{2\hbar_3+1} \in (0,1]$, and p is several positive integer.

Theorem 1: Let $\varsigma = \left\{\left(\mu, \alpha_\varsigma^\eta(\mu), \tau_\varsigma^\eta(\mu), \varphi_\varsigma'^\eta(\mu)\right) \mid \mu \in X\right\}$ and $\sigma = \{(\mu, \alpha_\sigma^\eta(\mu), \tau_\sigma^\eta(\mu), \varphi_\sigma'^\eta(\mu)) \mid \mu \in X\}$ be any two T-SFSs on universe X, then the SM among ς and σ is demarcated as $\beta_\hbar(\varsigma,\sigma)$, which placates the succeeding axioms:

(ϑ_1) $0 \le \beta_\hbar(\varsigma,\sigma) \le 1$;
(ϑ_2) $\beta_h(\varsigma,\sigma) = 1$ Iff $\varsigma = \sigma$;
(ϑ_3) $\beta_\hbar(\varsigma,\sigma) = \beta_\hbar(\sigma,\varsigma)$;
(ϑ_4) Let C be any TSFS such that $\varsigma \subseteq \sigma \subseteq C$, then $\beta_h(\varsigma, C) \le \beta_h(\varsigma,\sigma)$ and $\beta_h(\varsigma, C) \le \beta_h(\sigma, C)$.

Proof: In instruction to ascertain that Eq. (15) is a SM, we only need to demonstrate that Eq. (15) satisfies axioms $(\beta_1)-(\beta_4)$ $\varsigma = \{(\mu_\tau, \alpha_\varsigma(\mu_\tau), \tau_\varsigma(\mu_\tau), \varphi'{}_\varsigma(\mu_\tau)) \mid \mu_\tau \in X\}$ and $\sigma = \{((\mu_\tau, \alpha_\sigma(\mu_\tau), \tau_\sigma(\mu_\tau), \varphi'{}_\sigma(\mu_\tau))) \mid \mu_\tau \in X\}$ and $C = \{((\mu_\tau, \alpha_C(\mu_\tau), \tau_C(\mu_\tau), \varphi'_C(\mu_\tau))) \mid \mu_\tau \in X\}$ be any three T-SFSs on $X = \{\mu_1, \mu_2, \ldots, \mu_\eta\}$.

(ϑ_1) We can write the following equations:

$$\begin{aligned}\Delta_{1\varsigma\varsigma\sigma}(\mu_\tau) =& \frac{1}{\hbar_1+1}\left|\hbar_1\left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) - \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi'_\sigma(\mu_\tau)\right)\right| \hbar_1 \in [0,+\infty) \\ =& \frac{1}{\hbar_1+1}\left|\left(\hbar_1\alpha_\varsigma^\eta(\mu_\tau) - \tau_\varsigma^\eta(\mu_\tau) - \varphi_\varsigma'^\eta(\mu_\tau)\right) - \left(\hbar_1\alpha_\sigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau) - \varphi'_\sigma(\mu_\tau)\right)\right|\end{aligned}$$

$$\begin{aligned}\Delta_{2\varsigma\varsigma\sigma}(\mu_\tau) =& \frac{1}{2\hbar_2+1}\left|\hbar_2\left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right) - \left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) + \left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right)\right| \hbar_2 \in [0,+\infty) \\ =& \frac{1}{2\hbar_2+1}\left|\left(\hbar_2\tau_\varsigma^\eta(\mu_\tau) - \alpha_\varsigma^\eta(\mu_\tau) - \varphi_\varsigma'^\eta(\mu_\tau)\right) - \left(\hbar_2\tau_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau) - \varphi'_\sigma(\mu_\tau)\right)\right|\end{aligned}$$

$$\begin{aligned}\Delta_{3\varsigma\varsigma\sigma}(\mu_\tau) =& \frac{1}{2\hbar_3+1}\left|\hbar_3\left(\varphi_\varsigma'^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right) - \left(\alpha_\varsigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau)\right) + \left(\tau_\varsigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right)\right| \hbar_3 \in [0,+\infty) \\ =& \frac{1}{2\hbar_3+1}\left|\left(\hbar_3\varphi_\varsigma''(\mu_\tau) - \alpha_\varsigma^\eta(\mu_\tau) - \tau_\varsigma^\eta(\mu_\tau)\right) - \left(\hbar_3\varphi_\varsigma'^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau)\right)\right|\end{aligned}$$

By $\alpha_\varsigma^\eta(\mu_\tau), \tau_\varsigma^\eta(\mu_\tau), \varphi_\varsigma'^\eta(\mu_\tau), \alpha_\sigma^\eta(\mu_\tau), \tau_\sigma^\eta(\mu_\tau), \varphi_\sigma'^\eta(\mu_\tau) \in [0,1]$ and $\alpha_\varsigma^\eta(\mu_\tau) + \tau_\varsigma^\eta(\mu_\tau) + \varphi_\varsigma'^\eta(\mu_\tau) \le 1, \alpha_\sigma^\eta(\mu_\tau) + \tau_\sigma^\eta(\mu_\tau) + \varphi'^\eta(\mu_\tau) \le 1$

We have

$$\begin{gathered}-1 \le \hbar_1\alpha_\varsigma^\eta(\mu_\tau) - \tau_\varsigma^\eta(\mu_\tau) - \varphi_\varsigma'^\eta(\mu_\tau) \le \hbar_1 \\ -\hbar_1 \le -\left(\hbar_1\alpha_\sigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right) \le 1\end{gathered}$$

$0 \le \left|\left(\hbar_1\alpha_\varsigma^\eta(\mu_\tau) - \tau_\varsigma^\eta(\mu_\tau) - \varphi_\varsigma'^\eta(\mu_\tau)\right) - \left(\hbar_1\alpha_\sigma^\eta(\mu_\tau) - \tau_\sigma^\eta(\mu_\tau) - \varphi_\sigma'^\eta(\mu_\tau)\right)\right| \le \hbar_1 + 1$

i.e., $0 \le \Delta_{1\varsigma\sigma}(\mu_\tau) \le 1$,

then

$$\begin{gathered}-1 \le \hbar_2\tau_\varsigma^\eta(\mu_\tau) - \alpha_\varsigma^\eta(\mu_\tau) + \varphi_\varsigma'^\eta(\mu_\tau) \le 1 \vee \hbar_2 \\ -(1 \vee \hbar_2) \le -\left(\hbar_2\tau_\sigma^\eta(\mu_\tau) - \alpha_\sigma^\eta(\mu_\tau) + \varphi_\sigma'^\eta(\mu_\tau)\right) \le 1\end{gathered}$$

Similarly, we get the following inequalities:

$$
\begin{aligned}
-1 \leq \hbar_3 \varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\alpha_{\varsigma}^\eta\left(\mu_\tau\right)+\tau_{\varsigma}^\eta\left(\mu_\tau\right) \leq 1 \vee \hbar_3 \\
-\left(1 \vee \hbar_3\right) \leq-\left(\hbar_3 \varphi_\sigma^{\prime\prime \eta}\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)+\tau_\sigma^\eta\left(\mu_\tau\right)\right) \leq 1
\end{aligned}
$$

Then we obtain:

$$
\begin{aligned}
0 \leq\left|\left(\hbar_2 \tau_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_{\varsigma}^\eta\left(\mu_\tau\right)+\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)\right)-\left(\hbar_2 \tau_\sigma^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)+\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)\right| \leq 2 \vee \hbar_2 \\
0 \leq\left|\left(\hbar_3 \varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\alpha_{\varsigma}^\eta\left(\mu_\tau\right)+\tau_{\varsigma}^\eta\left(\mu_\tau\right)\right)-\left(\hbar_3 \varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)+\tau_\sigma^\eta\left(\mu_\tau\right)\right)\right| \leq 2 \vee \hbar_3
\end{aligned}
$$

It means that:

$$
\begin{aligned}
0 \leq \Delta_{2 \varsigma \sigma}\left(\mu_\tau\right)= & \frac{1}{2 \hbar_2+1}\left|\left(\hbar_2 \tau_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_{\varsigma}^\eta\left(\mu_\tau\right)+\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)\right)-\left(\hbar_2 \tau_\sigma^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)+\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)\right| \\
& \leq \frac{1}{\hbar_2+1} \vee \frac{1}{2} \leq 1 \\
0 \leq \Delta_{3 \varsigma \sigma}\left(\mu_\tau\right)= & \frac{1}{2 \hbar_3+1}\left|\left(\hbar_3 \varphi_{\varsigma}^{\prime}\left(\mu_\tau\right)-\alpha_{\varsigma}^\eta\left(\mu_\tau\right)+\tau_{\varsigma}^\eta\left(\mu_\tau\right)\right)-\left(\hbar_3 \varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)+\tau_\sigma^\eta\left(\mu_\tau\right)\right)\right| \\
& \leq \frac{1}{\hbar_3+1} \vee \frac{1}{2} \leq 1
\end{aligned}
$$

Finally, we have:

$$
0 \leq 1-\left[\frac{1}{3 \eta} \sum_{\tau=1}^\eta \Delta_{1 \varsigma \sigma}^p\left(\mu_\tau\right)+\Delta_{2 \varsigma \sigma}^p\left(\mu_\tau\right)+\Delta_{3 \varsigma \sigma}^p\left(\mu_\tau\right)\right]^{\frac{1}{p}} \leq 1
$$

Therefore,

$$
\left(\beta_1\right) 0 \leq \beta_{\hbar}(\varsigma, \sigma) \leq 1
$$

(ϑ_2) If $\varsigma=\sigma$ then $\alpha_{\varsigma}^\eta\left(\mu_\tau\right)=\alpha_\sigma^\eta\left(\mu_\tau\right), \tau_{\varsigma}^\eta\left(\mu_\tau\right)=\tau_\sigma^\eta\left(\mu_\tau\right)$ and $\varphi_{\varsigma}^{\prime}\left(\mu_\tau\right)=\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)$ Therefore, $\Delta_{1 \varsigma \sigma}\left(\mu_\tau\right)=0$, $\Delta_{2 \varsigma \sigma}\left(\mu_\tau\right)=0 \quad a \eta \varphi \Delta_{3 \varsigma \sigma}\left(\mu_\tau\right)=0$

i.e., $\quad \beta_{\hbar}(\varsigma, \sigma)=1$

If $\beta_{\hbar}(\varsigma, \sigma)=1$,

then

$$
\begin{aligned}
\Delta_{1 \varsigma \sigma}\left(\mu_\tau\right) & =\frac{1}{\hbar_1+1}\left|\hbar_1\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)\right|=0 \\
\Delta_{2 \varsigma \sigma}\left(\mu_\tau\right) & =\frac{1}{2 \hbar_2+1}\left|\hbar_2\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)+\left(\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)\right|=0 \\
\Delta_{3 \varsigma}\left(\mu_\tau\right) & =\frac{1}{2 \hbar_3+1}\left|\hbar_3\left(\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)-\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)+\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)\right|=0
\end{aligned}
$$

By definition, absolute value gives us:

$$
\begin{aligned}
& \hbar_1\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\varphi_{\varsigma}^{\prime}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)=0 \\
& \hbar_2\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)-\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)+\left(\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)\right)=0 \\
& \hbar_3\left(\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime\prime}\left(\mu_\tau\right)\right)-\left(\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right)\right)+\left(\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right)\right)=0
\end{aligned}
$$

i.e.,

$$
\left(\begin{array}{ccc}
\hbar_1 & -1 & -1 \\
-1 & \hbar_2 & 1 \\
-1 & 1 & \hbar_3
\end{array}\right)\left(\begin{array}{c}
\alpha_{\varsigma}^\eta\left(\mu_\tau\right)-\alpha_\sigma^\eta\left(\mu_\tau\right) \\
\tau_{\varsigma}^\eta\left(\mu_\tau\right)-\tau_\sigma^\eta\left(\mu_\tau\right) \\
\varphi_{\varsigma}^{\prime \eta}\left(\mu_\tau\right)-\varphi_\sigma^{\prime \eta}\left(\mu_\tau\right)
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right)
$$

Since $\frac{1}{\hbar_1+1}+\frac{1}{2 \hbar_2+1}+\frac{1}{2 \hbar_3+1} \in(0,1]$ then $2 \leq \hbar_1 \hbar_2 \hbar_3-\left(\hbar_1+\hbar_3+\hbar_3\right)$.

By using the matrix determinant definition, we can obtain:

$$
\left|\begin{array}{ccc}
\hbar_1 & -1 & -1 \\
-1 & \hbar_2 & 1 \\
-1 & 1 & \hbar_3
\end{array}\right|=\hbar_1 \hbar_2 \hbar_3+2-\left(\hbar_1+\hbar_3+\hbar_3\right) \geq 4
$$

Therefore, we have

$$\left(\begin{array}{c}\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\\ \tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\\ \varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\end{array}\right)=\left(\begin{array}{ccc}\hbar_{1}&-1&-1\\ -1&\hbar_{2}&1\\ -1&1&\hbar_{3}\end{array}\right)^{-1}\left(\begin{array}{c}0\\ 0\\ 0\end{array}\right)=\left(\begin{array}{c}0\\ 0\\ 0\end{array}\right)$$

It means that $\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)=\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right),\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)=\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)$ and $\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)=\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)$ then $\varsigma=\sigma$

(ϑ_{3}) From the definition of absolute value, the following equations can be obtained:

$$\begin{aligned}\Delta_{1\varsigma\sigma}\left(\mu_{\tau}\right)&=\frac{1}{\hbar_{1}+1}\left|\hbar_{1}\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\frac{1}{\hbar_{1}+1}\left|(-1)\left[\hbar_{1}\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\varphi_{\sigma}^{\prime}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right]\right|\\ &=\frac{1}{\hbar_{1}+1}\left|\hbar_{1}\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\Delta_{1\sigma\varsigma}\left(\mu_{\tau}\right)\end{aligned}$$

$$\begin{aligned}\Delta_{2\varsigma\sigma}\left(\mu_{\tau}\right)&=\frac{1}{2\hbar_{2}+1}\left|\hbar_{2}\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\frac{1}{2\hbar_{2}+1}\left|(-1)\left[\hbar_{2}\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\zeta}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right]\right|\\ &=\frac{1}{2\hbar_{2}+1}\left|\hbar_{2}\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\Delta_{2\sigma\varsigma}\left(\mu_{\tau}\right)\end{aligned}$$

$$\begin{aligned}\Delta_{3\varsigma\sigma}\left(\mu_{\tau}\right)&=\frac{1}{2\hbar_{3}+1}\left|\hbar_{3}\left(\varphi_{\varsigma}^{\prime}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\frac{1}{2\hbar_{3}+1}\left|(-1)\left[\mid\hbar_{3}\left(\varphi_{\sigma}^{\prime}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\prime}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)\right]\right|\\ &=\frac{1}{2\hbar_{3}+1}\left|\hbar_{3}\left(\varphi_{\sigma}^{\prime\prime}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)-\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)\right|\\ &=\Delta_{3\sigma\varsigma}\left(\mu_{\tau}\right)\end{aligned}$$

$$\beta_{\hbar}(\varsigma,\sigma)=\beta_{\hbar}(\sigma,\varsigma)$$

(ϑ_{4}) Therefore $\varsigma\subseteq\sigma\subseteq C$ then $\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\leq\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\leq\alpha_{C}^{\eta}\left(\mu_{\tau}\right),\tau_{C}^{\eta}\left(\mu_{\tau}\right)\leq\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\leq\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right),\varphi_{C}^{\prime\eta}\left(\mu_{\tau}\right)\leq\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\leq\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)$

Thus, we can get

$$\begin{aligned}&\hbar_{1}\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{1}\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{1}\left(\alpha_{C}^{\eta}\left(\mu_{\tau}\right)-\tau_{c}^{\eta}\left(\mu_{\tau}\right)-\varphi_{C}^{\prime\eta}\left(\mu_{\tau}\right)\right)\\ &\hbar_{2}\left(\tau_{c}^{\eta}\left(\mu_{\tau}\right)-\alpha_{C}^{\eta}\left(\mu_{\tau}\right)-\varphi_{c}^{\prime\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{2}\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{2}\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\\ &\hbar_{3}\left(\varphi_{c}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{C}^{\eta}\left(\mu_{\tau}\right)-\tau_{C}^{\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{3}\left(\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)\leq\hbar_{3}\left(\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)\end{aligned}$$

Using the inequality property, we can get:

$$\begin{aligned}&\left|\hbar_{1}\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)-\hbar_{1}\left(\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ \leq&\left|\hbar_{1}\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)-\hbar_{1}\left(\alpha_{c}^{\eta}\left(\mu_{\tau}\right)-\tau_{c}^{\eta}\left(\mu_{\tau}\right)-\varphi_{C}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &\left|\hbar_{2}\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)-\hbar_{2}\left(\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ \leq&\left|\hbar_{2}\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)-\hbar_{2}\left(\tau_{c}^{\eta}\left(\mu_{\tau}\right)-\alpha_{c}^{\eta}\left(\mu_{\tau}\right)-\varphi_{c}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|\\ &\left|\hbar_{3}\left(\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\hbar_{3}\left(\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)\right|\\ \leq&\left|\hbar_{3}\left(\varphi_{\varsigma}^{\prime}\left(\mu_{\tau}\right)-\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)\right)-\hbar_{3}\left(\varphi_{c}^{\prime\eta}\left(\mu_{\tau}\right)-\alpha_{C}^{\eta}\left(\mu_{\tau}\right)-\tau_{C}^{\eta}\left(\mu_{\tau}\right)\right)\right|\end{aligned}$$

$$\Delta_{2\varsigma\sigma}\left(\mu_{\tau}\right)\leq\Delta_{1\varsigma c}\left(\mu_{\tau}\right),\Delta_{2\varsigma\sigma}\left(\mu_{\tau}\right)\leq\Delta_{2\varsigma c}\left(\mu_{\tau}\right),\Delta_{3\varsigma\sigma}\left(\mu_{\tau}\right)\leq\Delta_{3\varsigma c}\left(\mu_{\tau}\right)$$

Consequently, we have

$$1-\left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta}\Delta_{1\varsigma c}^{p}\left(\mu_{\tau}\right)+\Delta_{2\varsigma c}^{p}\left(\mu_{\tau}\right)+\Delta_{3\varsigma c}^{p}\left(\mu_{\tau}\right)\right]^{\frac{1}{p}}=1-\left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta}\Delta_{1\varsigma\sigma}^{p}\left(\mu_{\tau}\right)+\Delta_{2\varsigma\sigma}^{p}\left(\mu_{\tau}\right)+\Delta_{3\varsigma\sigma}^{p}\left(\mu_{\tau}\right)\right]^{\frac{1}{p}}$$

It resources that $\beta_{\hbar}(\varsigma, C) \leq \beta_{\hbar}(\varsigma, \sigma)$.
Similarity, we have $\beta_{\hbar}(\varsigma, C) \leq \beta_{\hbar}(\sigma, C)$
(1) When $\hbar_1 = 0, \hbar_2 = \hbar_3 = +\infty$, Eq. (16) can be written as:

$$\beta_1(\varsigma,\sigma)=1-\left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta}\begin{pmatrix}\left|\left(\tau_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\tau_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)+\left(\varphi_{\varsigma}^{\prime\eta}\left(\mu_{\tau}\right)-\varphi_{\sigma}^{\prime\eta}\left(\mu_{\tau}\right)\right)\right|^{p}+\\ \frac{|(\tau_{\varsigma}^{\eta}(\mu_{\tau})-\tau_{\sigma}^{\eta}(\mu_{\tau}))|p}{2^{p}}+\frac{|\left(\varphi_{\varsigma}^{\eta}(\mu_{\tau})-\varphi_{\sigma}^{\prime\eta}(\mu_{\tau})\right)|p}{2^{p}}\end{pmatrix}\right]^{\frac{1}{p}} \tag{16}$$

(2) When $\hbar_1 = \hbar_2 = +\infty \hbar_3 = 0$, Eq. (17) can be written as:

$$\beta_2(\varsigma,\sigma)=1-\left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta}\begin{pmatrix}\left|\left(\alpha_{\varsigma}^{\eta}\left(\mu_{\tau}\right)-\alpha_{\sigma}^{\eta}\left(\mu_{\tau}\right)\right)\right|^{p}+\\ \frac{|(\tau_{\varsigma}^{\eta}(\mu_{\tau})-\tau_{\sigma}^{\eta}(\mu_{\tau}))|^{p}}{2^{p}}+\frac{|((\tau_{\varsigma}^{\eta}(\mu_{\tau})-\tau_{\sigma}^{\eta}(\mu_{\tau})))-(\alpha_{\varsigma}^{\eta}(\mu_{\tau})-\alpha_{\sigma}^{\eta}(\mu_{\tau}))|^{p}}{2^{p}}\end{pmatrix}\right]^{\frac{1}{p}} \tag{17}$$

Theorem 2: For any two T-SFSs $\varsigma = \left\{\left(\mu_{\tau}, \alpha_{\varsigma}\left(\mu_{\tau}\right), \tau_{\varsigma}\left(\mu_{\tau}\right), \varphi_{\varsigma}^{\prime}\left(\mu_{\tau}\right)\right) \mid \mu_{\tau} \in X\right\}$ and

$$\sigma=\{((\mu_{\tau},\alpha_{\sigma}\left(\mu_{\tau}\right),\tau_{\sigma}\left(\mu_{\tau}\right),\varphi'{}_{\sigma}\left(\mu_{\tau}\right)))\mid\mu_{\tau}\in X\}$$

on $X = \{\mu_1, \mu_2, \ldots, \mu_\eta\}\, \omega_\tau \in [0,1]$ and $\sum_{\tau=1}^{\eta} \omega_\tau = 1$ the function $\beta_\omega : T - \mathrm{SFS}(\mu) \times T - \mathrm{SFS}(\mu) \to [0,1]$ is defined by

$$\beta_{\omega}(\varsigma,\sigma)=1-\left[\frac{1}{3\eta}\sum_{\tau=1}^{\eta}\omega_{\tau}\left(\Delta_{1\varsigma\sigma}^{p}\left(\mu_{\tau}\right)+\Delta_{2\varsigma\sigma}^{p}\left(\mu_{\tau}\right)+\Delta_{3\varsigma\sigma}^{p}\left(\mu_{\tau}\right)\right)\right]^{\frac{1}{p}}$$

$\beta_\omega(\varsigma, \sigma)$ is a weighted SM between ς and σ.
Proof: Theorem 2 and the proof are comparable.
To further clarify, an example is added in the following.

Example 1: Let $\varsigma = (\mu, 0.1, 0.3, 0.4), \sigma = (\mu, 0.5, 0.2, 0.1)$ and C $= (\mu, 0.5, 0.0, 0.0)$ are three different T-SFVs on X $= \{\mu\}$. A is more similar to B than C to say $\beta(\varsigma, \sigma) > \beta(\varsigma, C)$. To demonstrate this view's accuracy for our suggested SM $\beta_{\hbar}$, and the existing ones in particular $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\beta_7, \beta_9, \beta_{10}\beta_{11}\beta_{12}, \beta_{13}, \beta_{14}, \beta_{\hbar}$. Table 2 displays the SMs' obtained values.

Table 2. The SMs' values on T-SFVs ς, σ, and C

$\boldsymbol{\beta}$	$\boldsymbol{\beta(\varsigma, B)}$	$\boldsymbol{\beta(\varsigma, C)}$	**Relation**
$\boldsymbol{\beta_1}$	0.9587	0.9587	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_2}$	0.9657	0.9642	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_3}$	0.9622	0.9614	$\beta\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_4}$	0.9587	0.9587	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_5}$	0.7402	0.7402	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_6}$	0.7794	0.7794	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_7}$	0.0154	0.0015	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_8}$	1.000	1.000	$\beta(\varsigma,\sigma)$=$\beta(C,\sigma)$
$\boldsymbol{\beta_9}$	0.3009	0.2996	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_{10}}$	0.9026	0.8988	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_{11}}$	0.0155	0.0048	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_{12}}$	0.0078	0.0024	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_{13}}$	0.0086	0.0027	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_{14}}$	0.0043	0.0013	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$
$\boldsymbol{\beta_m}$	0.6000	0.4500	$\beta(\varsigma,\sigma)$>$\beta(C,\sigma)$

4 Using the Suggested Similarity Metrics

This section applies the suggested SMs to MADM problems, demonstrating that the expected SM makes sense and is consistent with human cognition.

Let $X = \{\mu_1, \mu_2, \ldots \mu_\hbar\}$ be a set of attributes, the η alternatives

$$\varsigma_\tau = (\varsigma_{\tau j}) = \{(\mu_j, \alpha_{\varsigma\tau}(\mu_j), \tau_{\varsigma\tau}(\mu_j), \varphi'_{\varsigma\tau}(\mu_j))\} \mid \mu_j \in X,$$

where, $\mu_j, \alpha^\eta_\varsigma(\mu_\tau), \tau^\eta_\varsigma(\mu_\tau), \varphi'^\eta_\varsigma(\mu_\tau), \alpha^\eta_\sigma(\mu_\tau), \tau^\eta_\sigma(\mu_\tau), \varphi'^\eta_\sigma(\mu_\tau) \in [0,1]$ and $\alpha^\eta_\varsigma(\mu_\tau) + \tau^\eta_\varsigma(\mu_\tau) + \varphi'^\eta_\varsigma(\mu_\tau) \leq 1, \alpha^\eta_\varsigma(\mu_\tau)$ is a positive degree that is used to determine whether the alternative ς_τ satisfies the $\mu_j(\tau = \{1, 2, \ldots \eta\}, j = \{1, 2, \ldots \hbar\}), \tau^\eta_\varsigma(\mu_\tau)$.

A neutral degree, $\tau^\eta_\varsigma(\mu_\tau)$, is used to ensure that ς_τ does not satisfy the μ_j. A negative degree $\varphi'(\mu_\tau)$ is used to ensure that ς_τ does not satisfy the μ_j. The best course of action is selected through decision-making.

Step 1 Standardize the options for making decisions.

There are two categories of multi-attribute decision-making in this process: amount type and interest type.

The formula used in the decision-making process can be used to convert the amount type into the interest type.

$$\varsigma'_{\tau j} = \begin{cases} \varsigma_{\tau j} \text{ for benefit attribute } \mu_j \\ \varsigma^c_{\tau j} \text{ for cost attribute } \mu_j \end{cases}$$

$\varsigma^c_{\tau j} = (\alpha_{\varsigma\tau}(\mu_j), \tau_{\varsigma\tau}(\mu_j), \varphi'_{\varsigma\tau}(\mu_j)), \tau = \{1, 2, \ldots \eta\}, j = \{1, 2 \ldots \hbar\}$. The above formula is based on the alternative $_\tau = \{\varsigma'_{\tau j}\}$.

Step 2 The SM $\beta = (\varsigma_\tau, \varsigma)\ (\tau = 1, 2, 3, \ldots, \eta)$ is calculated, where $\varsigma = \{(0.11, 0.21, 0.32), (0.11, 0.21, 0.32), (0.11, 0.21, 0.32)\}$ is a standard that the T-SFV, the decision-maker, provides. With the aid of the suggested SM, we determine the similarity values.

Step 3 The maximum one is chosen in $\beta = (Doc_{\tau 0}, \varsigma)$ from $\beta = (\varsigma_\tau, \varsigma)\, \tau = (c = 1, 2, 3, \ldots \eta))\tau.e$

$\beta = (Doc_{\tau 0}, \varsigma) = \underset{1 \leq \tau \leq \eta}{\mathrm{ha}\mu} \{\beta(Doc_\tau, \varsigma)\}$. The maximum SMs alternative $Doc_{\tau 0}$ then follows the maximum principle.

For the similarity measure, see the example below $\beta_\hbar, p = 3, \hbar_1 = \hbar_2 = \hbar_3 = 3$

4.1 Example

In the current digital era, internet skills are crucial. These include the capacity to use online platforms, critically assess information, and effectively communicate. Online guides and discussion boards, as well as official educational programs provided by institutions and organizations, are good places to start when trying to hone these abilities. Learning how to use the Internet effectively can help people in many ways. They can connect with different communities, access a plethora of information, and use online resources for personal growth, career advancement, and education. Accepting the potential of the Internet enables people to prosper in a world that is becoming more interconnected. There are three alternative ς_1 Web Savvy, ς_2 Digital Literacy, ς_3 Cyber Skills, with three different attributes, x_1 Information Fluency, x_2 Online Communication, x_3 Cyber security, labelled the T-SFSs as presented in Table 3. The weights of $\mu_j (1 \leq j \leq 3)$ are $(0.5, 0.3, 0.2)$.

Table 3. Three options, each with three characteristics

	x_1	x_2	x_3
ς_1	(0.12,0.13,0.16)	(0.16,0.14,0.28)	(0.23,0.12,0.11)
ς_2	(0.15,0.14,0.11)	(0.21,0.16,0.20)	(0.32,0.0,0.11)
ς_3	(0.20,0.08,0.07)	(0.21,0.12,0.11)	(0.18,0.0,0.07)

Table 4 below gives the values of the SMs of ς_1, ς_2, and ς_3 by ς.

Based on the attributes, Table 4 displays the SMs values for the Internet skills. Based on the values derived from the SMs, we can now determine the ranking of the alternatives. Table 5 below displays the Internet skills decision-ranking.

The alternative ς_1 is obtained by using the SMs $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}$ and $\beta_\hbar$. After it is cleared from Table 5. On the other hand, the β_8 yields the alternative ς_1. Figure 1 provides the following geometric representation of the order of the Internet skills and in Doc formula β_8 not given the answer.

It is cleared from Figure 1, the alternative ς_1 is obtained by using the SMs $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}$ and β_h. However, the alternative ς_1 is obtained by the β_8.

Table 4. Values of the example's similarity metrics and decision outcomes 2

	$\beta(\varsigma_1,\varsigma)$	$\beta(\varsigma_2,\varsigma)$	$\beta(\varsigma_3,\varsigma)$
β_1	0.9693	0.9638	0.9594
β_2	0.9823	0.975	0.9765
β_3	0.7192	0.6725	0.6793
β_4	0.9764	0.9708	0.9679
β_5	0.046	0.0568	0.0622
β_6	0.9338	0.9194	0.9122
β_7	0.2732	0.1388	0.0548
β_8	0.7035	0.1687	0.9439
β_9	0.9823	0.9752	0.9766
β_{10}	0.9872	0.9807	0.9765
β_{11}	0.6567	0.3698	0.0961
β_{12}	0.2554	0.1413	0.0335
β_{13}	0.264	0.1089	0.0234
β_{14}	0.087	0.0356	0.0082
β_m	0.9882	0.9833	0.9843

Table 5. Using the values of the SMs found in table to rank the Internet skills

	Ranking	**The Best Alternative**	**Doc**
β_1	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.1352
β_2	$\varsigma_1 > \varsigma_3 > \varsigma_2$	ς_1	-0.03
β_3	$\varsigma_1 > \varsigma_3 > \varsigma_2$	ς_1	0.248
β_4	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.0656
β_5	$\varsigma_3 > \varsigma_2 > \varsigma_1$	ς_3	0.095
β_6	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.0843
β_7	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.2639
β_8	$\varsigma_3 > \varsigma_1 > \varsigma_2$	ς_3	0
β_9	$\varsigma_1 > \varsigma_3 > \varsigma_2$	ς_1	-0.0416
β_{10}	$\varsigma_1 > \varsigma_3 > \varsigma_2$	ς_1	-0.038
β_{11}	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	0.1057
β_{12}	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	0.0149
β_{13}	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.9751
β_{14}	$\varsigma_1 > \varsigma_2 > \varsigma_3$	ς_1	-0.4779
β_m	$\varsigma_1 > \varsigma_3 > \varsigma_2$	ς_1	-0.0189

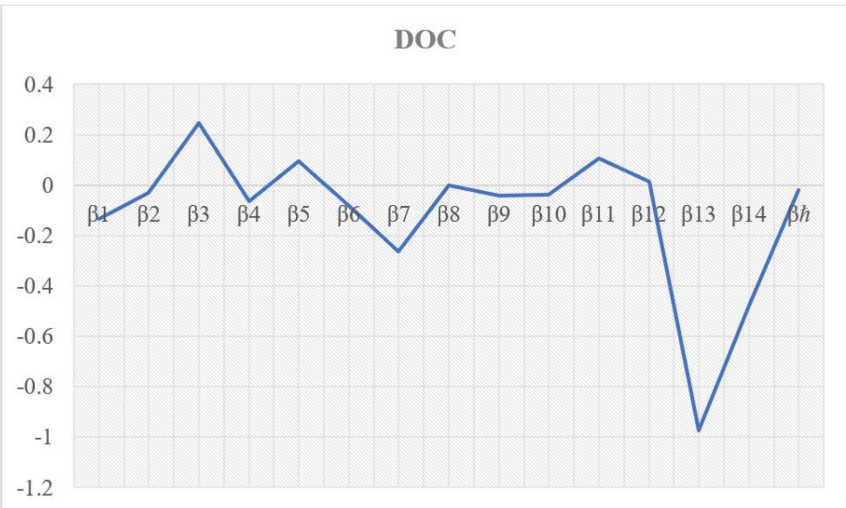

Figure 1. Table 5 lists the Internet skills in order of acquisition from the SMs

4.2 Example

In this example, a problem with the MADM is solved. The Internet has completely changed how people can obtain information, allowing them to easily research a wide range of topics on almost any subject imaginable. People can perform academic research, pursue self-directed learning, and rapidly find answers to their questions using search engines, online libraries, and educational websites. The Internet also makes the world more connected, which promotes cooperation and communication between people in different places. The Internet allows people to connect with like-minded people worldwide, follow their passions, and broaden their horizons, whether for personal enrichment, professional development, or academic research.

ς_1 Online Advantage

ς_2 Digital Asset

ς_3 Cyber Advantage

ς_4 Virtual Benefit

ς_5 Web Advantage

ς_6 Online Benefit

Six candidates will be evaluated based on these characteristics. Take into consideration $\varsigma = \{(0.11, 0.21, 0.32), (0.11, 0.21, 0.32), (0.11, 0.21, 0.32)\}$, where the T-SFV is used as the standard. The option that most closely resembles S is regarded as the ideal worker. Consider some attributes of internet benefit i.e., Global Access (x_1), Information Accessibility (x_2) and Connectivity (x_3). Weight vector $\omega = (0.2, 0.3, 0.5)$ is what it is. Following an initial evaluation, the T-SFVs are assigned to the employees based on the characteristics listed in Table 6.

Table 6. In this case, the evaluation outcomes of six faculty candidate

	x_1	x_2	x_3
ς_1	(0.22,0.31,0.40)	(0.53,0.03,0.21)	(0.11,0.31,0.07)
ς_2	(0.0,0.30,0.54)	(0.32,0.34,0.20)	(0.36,0.43,0.20)
ς_3	(0.41,0.34,0.15)	(0.12,0.04,0.32)	(0.12,0.02,0.43)
ς_4	(0.23,0.55,0.20)	(0.55,0.34,0.10)	(0.31,0.31,0.1)
ς_5	(0.33,0.21,0.23)	(0.21,0.43,0.21)	(0.23,0.33,0.0)
ς_6	(0.27,0.12,0.16)	(0.22,0.32,0.28)	(0.33,0.0,0.12)

Next, using the existing and proposed SMs, each candidate's similarity to the standard is assessed. The outcomes are summarized as follows in Table 7.

Table 7. In this example, similarity metrics and decision outcomes

	$\beta(\varsigma_1, \varsigma)$	$\beta(\varsigma_2, \varsigma)$	$\beta(\varsigma_3, \varsigma)$	$\beta(\varsigma_4, \varsigma)$	$\beta(\varsigma_5, \varsigma)$	$\beta(\varsigma_6, \varsigma)$
β_1	0.9197	0.8935	0.9487	0.8627	0.9499	0.9636
β_2	0.9609	0.9717	0.9811	0.9453	0.9815	0.9779
β_3	0.5951	0.6623	0.7931	0.4979	0.7168	0.6932
β_4	0.9404	0.9268	0.9619	0.8926	0.965	0.9753
β_5	0.1268	0.138	0.0781	0.2055	0.0872	0.0562
β_6	0.8256	0.8154	0.8901	0.7386	0.879	0.9201
β_7	0.214	0.1735	0.4189	0.0874	0.1954	0.1941
β_8	0.084	0.9488	0.6437	0.6073	0.4668	0.3395
β_9	0.9527	0.9382	0.9682	0.9194	0.9621	0.9713
β_{10}	0.9292	0.9405	0.9678	0.899	0.9641	0.9802
β_{11}	0.3508	0.4088	0.7103	0.1163	0.1705	0.2733
β_{12}	0.1703	0.1861	0.195	0.0554	0.0778	0.0975
β_{13}	0.1641	0.1138	0.4755	0.0286	0.1323	0.209
β_{14}	0.0801	0.0472	0.1349	0.0123	0.0629	0.0714
β_m	0.97393	0.9811	0.9873	0.9635	0.9876	0.9852

With the aid of the current and proposed SMs, Table 7 displays the values of the staff members' SMs based on the attributes. Table 8 in the following provides the ranking of the Internet benefits.

It is unoccupied from Table 8, the candidate ς_6 is attained by using the SMs $\beta_1, \beta_4, \beta_9, \beta_{10}$, the candidate ς_5 is obtained by using the SMs β_2 and $\beta_\hbar$, the candidate ς_3 is obtained by using the SMs $\beta_3, \beta_7, \beta_{11}, \beta_{12}, \beta_{13}$, and β_{14}, and the candidate ς_4 is obtained by using the SMs β_5, and the candidate ς_2 is obtained by using the SMs β_8.

Table 8. A comparison of the Internet benefits that the proposed and current SMs have acquired

	Ranking	The best candidate	Doc
β_1	$\varsigma_6 > \varsigma_5 > \varsigma_3 > \varsigma_1 > \varsigma_2 > \varsigma_4$	ς_6	-0.0199
β_2	$\varsigma_5 > \varsigma_3 > \varsigma_6 > \varsigma_2 > \varsigma_1 > \varsigma_4$	ς_5	-0.053
β_3	$\varsigma_3 > \varsigma_5 > \varsigma_5 > \varsigma_2 > \varsigma_1 > \varsigma_4$	ς_3	-0.3878
β_4	$\varsigma_6 > \varsigma_5 > \varsigma_3 > \varsigma_1 > \varsigma_2 > \varsigma_4$	ς_6	0.1492
β_5	$\varsigma_4 > \varsigma_2 > \varsigma_1 > \varsigma_5 > \varsigma_3 > \varsigma_6$	ς_4	-0.8148
β_6	$\varsigma_6 > \varsigma_3 > \varsigma_5 > \varsigma_1 > \varsigma_2 > \varsigma_4$	ς_6	0.6968
β_7	$\varsigma_3 > \varsigma_1 > \varsigma_5 > \varsigma_6 > \varsigma_2 > \varsigma_4$	ς_3	-0.4705
β_8	$\varsigma_2 > \varsigma_3 > \varsigma_4 > \varsigma_5 > \varsigma_6 > \varsigma_1$	ς_2	-2.3613
β_9	$\varsigma_6 > \varsigma_3 > \varsigma_5 > \varsigma_1 > \varsigma_2 > \varsigma_4$	ς_6	0.3288
β_{10}	$\varsigma_6 > \varsigma_3 > \varsigma_5 > \varsigma_2 > \varsigma_1 > \varsigma_4$	ς_6	-0.106
β_{11}	$\varsigma_3 > \varsigma_2 > \varsigma_6 > \varsigma_1 > \varsigma_5 > \varsigma_4$	ς_3	-0.1827
β_{12}	$\varsigma_3 > \varsigma_2 > \varsigma_1 > \varsigma_6 > \varsigma_5 > \varsigma_4$	ς_3	-0.2756
β_{13}	$\varsigma_3 > \varsigma_6 > \varsigma_1 > \varsigma_5 > \varsigma_2 > \varsigma_4$	ς_3	-0.1387
β_{14}	$\varsigma_3 > \varsigma_1 > \varsigma_6 > \varsigma_5 > \varsigma_2 > \varsigma_4$	ς_3	0.0718
β_m	$\varsigma_5 > \varsigma_3 > \varsigma_6 > \varsigma_2 > \varsigma_1 > \varsigma_4$	ς_5	-0.0352

In Figure 2, the ranking is similarly shown geometrically as follows. In Doc formula we compare the parameters with each other. Doc formula is defined in step 3.

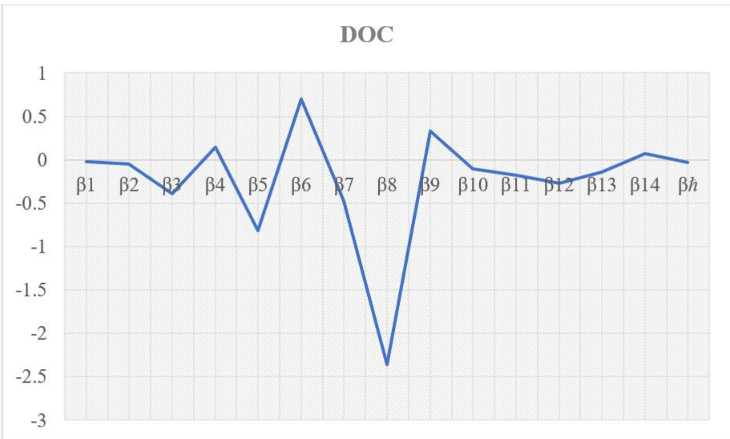

Figure 2. The ranking of the Internet benefit obtained from the SMs in Table 8

5 Conclusion

To assess the similarity between two T-SFVs, new SMs are defined for T-SFS in this study. The recently created SM for T-SFS, which expands on the parameters of the current SMs. The discussion and mathematical work demonstrate the adaptability and viability of the suggested SM. It has been discussed what the current SMs for T-SFS can and cannot do. The following advances are some of the ones that are discussed:

• The recommended SM fulfills the axiom (S2) of the SM, which makes the recommended SM stay away from the circumstances of counter-intuitive inferences $\varsigma = \sigma$ infers $\beta(\varsigma, \sigma) = 1$. But the proposed SMs satisfy this axiom.

• The proposed SMs are based parameters $\hbar_1, \hbar_2$ and $\hbar_3$. Hence, the decision-makers have a choice to select the values of these parameters individually. For this condition, decision-makers' cylinders take the proper boundaries $\hbar_1, \hbar_2, \hbar_3$ to get the sensible SM, which is in accordance with the ongoing leader style and choice climate.

• The proposed SM can give sensible dependable choice results to leaders. The proposed SM has the most significant level of believability, yet in addition, it can take care of the dynamic issues that the current SMs cannot settle and get sensible choice outcomes. Hence, the proposed SM is sensible and adaptable.

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Conflicts of Interest

The authors declare that none of the work reported in this paper could have been influenced by any known competing financial interests or personal relationships.

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