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The Optimal Tariff Policy and Trade Competitiveness with Endogenous Timing in a Differentiated Duopoly

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Abstract: Tariff is an effective means to protect domestic enterprises and improve the competitiveness of domestic products. This paper constructs a differentiated duopoly model considering endogenous timing to investigate the tariff policy and the impact of product differentiation on the equilibrium results. The conclusions are presented by analyzing the observable two-period delay game as follows. In quantity competition, the welfare-maximizing government sets the tariff level under the home-leading Stackelberg equilibrium, which is contained in the choices of the two firms in the subsequent observable delay game. In price competition, the Bertrand equilibrium is best for the government. However, either of the two Stackelberg equilibria is optimal in observable delay game. It suggests that adjusting the tariff level cannot sufficiently encourage the firms to adopt the welfare-maximizing duopoly determinately. Moreover, increasing product differentiation enhances the home social welfare in home-leading Stackelberg competition but reduces consumer surplus in Bertrand competition.

Keywords: Endogenous timing; Competitiveness; Differentiated duopoly; Tariff; Quantity and price competition

JEL Classification: L1; L13; L52

1. Introduction

Tariff, formulated by the home country, has become a global trade strategy, and thus it provides significant welfare implications for improving the competitiveness of domestic products. "Trade war" from rising global competition has become a hot topic in the media recently. It is widely accepted that tariff is one of the alternative sources and instruments of government revenue, exerting remarkable influence on social welfare (Collie, 2020). Therefore, the optimal tariff has become a matter of topical policy debate. Since Bickerdike (1906), the optimal tariff theory with strategic interdependence has been proposed and extended in various ways. For example, based on the theoretical and empirical results, Xu & Lee (2019) compare the strategic tariff for domestic welfare maximization to the efficient tariff for global welfare maximization by considering the corporate social responsibility level. Agreeing with that strategic trade interaction between two governments usually reduces welfare, Yu & Lee (2011) show that production subsidy as the optimal trade policy can yield a better social welfare benefit than the import tariff. Oda (2008) concludes that the impact on welfare depends on the tariff revenue effect and the repatriation effect in a developing country. Moreover, DeJong & Ripoll (2006) and Rojas-Vallejos & Turnovsky (2017) have empirically revealed that the tariff level negatively impacts income inequality and growth of developed countries. In a word, the literature in this field explores the implications of tariff policies under different frameworks and obtains diverse conclusions.

However, as the open global economy grows, endogenous timing in international trade, where domestic firms compete with foreign firms with alternative orders, has received increasing attention from scholars in recent years (Hayashibara, 2002; Pal, 1998; Toshimitsu, 2013). Theoretically, it is well-known that firms would prefer to participate in a sequential game where they move second as followers to one game where they are leaders. Such preferences exist in different and considerably more general settings than those just described. In detail, firms first choose the timing of their price or quantity decisions (two possibilities; "early" or "late"), and the relevant basic game is then played out. Again, it is well-known that such a timing game has multiple (two) pure strategy subgame perfect equilibria, each entailing sequential moves, in either order. Specifically, Stackelberg duopoly reveals the sequential nature of choices as an alternative to the simultaneity of moves in Cournot or Bertrand equilibrium. The use of Stackelberg equilibrium can date back to Edgeworth (2024) that criticizes the simultaneous game based on the possibility of either firm gaining from Stackelberg leadership. It is not wise to accept the exogeneity of timing difference imposed on firms, although describing the detailed characteristics of firm's competition process (Chen et al., 2021a; Gabszewicz & Thisse, 1979; Shaked & Sutton, 1983). Collie (1994) investigates the importing country's tariff policy game with the endogenous timing decision in a two-country duopoly. Hamada (2021) points out that the observable delay game in a mixed duopoly has multiple equilibria and the firms choose either of two Stackelberg equilibria. Hamilton & Slutsky (1990) show that firms consider whether to act at the first opportunity or to wait until observing the rivals' first period actions in the pre-play stage. The sequential order of move may give rise to highly different outcomes in a simultaneous game, which should be analysed endogenously (Robson, 1990). Different from the traditional literature on tariff policy focusing on simultaneous-move games (Chen et al., 2020a; Chen et al., 2020b; Gabszewicz & Thisse, 1980; Motta, 1993; Shaked & Sutton, 1982), endogenous timing should be introduced in competition model to explore the trade strategies. Generally, the home government sets the tariff to maximize social welfare, which seriously affects foreign firms' costs. Therefore, how does tariff policy affect endogenous timing choices? Whether the timing difference can induce more possible equilibria? Considering the product differentiation, how can the home government formulate tariff policy? To answer these questions, this paper examines the equilibria of the differentiated quantity and price duopolies where the timing of moves is endogenized with observable delay game of Hamilton & Slutsky (1990), and analyses the effect of product homogeneity on equilibrium results.

Based on the above analysis, we find that most existing literature on tariff policy constructs a single competition mode (quantity or price) (Bárcena-Ruiz, 2007; Fujiwara, 2007; Lin & Matsumura, 2012; Matsumura, 2003; Matsumura & Kanda, 2005; Merrill & Schneider, 1966; Ogawa & Kato, 2006) or ignores the role of the timing difference between trade partners (Chen et al., 2022; Soderbery, 2018; Wang, 2007; Yang & Nie, 2020). To make up for the above defects, the contributions of this article are as follows. On the one hand, our study transforms the setting of timing decisions exogenously given to a case in which the government endogenously sets the optimal tariff level depending on the objective function. We discuss the possible equilibrium when endogenizing the timing decision in the observable delay game. On the other hand, the price or quantity competition modes are both taken into consideration to further investigate the optimal strategies, which extends the interesting findings of tariff policy with endogenous timing. The main results of this study show that, in quantity competition, the welfare-maximizing home government has an incentive to decrease the tariff level for the foreign firm until realizing the Stackelberg equilibrium when the home firm is the leader, which is identical to the choice of the two firms in the subsequent observable delay game. In price competition, the home government prefers the Bertrand equilibrium. However, the firms choose either of the two Stackelberg equilibria in the observable delay game, which cannot cooperate with the goals of different market players. The results suggest that adjusting the tariff level cannot sufficiently encourage the firms to adopt the welfare-maximizing duopoly determinately. When the home government sets an appropriate tariff level for the foreign firm, the alternative possible equilibrium outcomes in the endogenous timing game are limited. Moreover, increasing product differentiation enhances the home social welfare in home-leading Stackelberg competition but reduces consumer surplus in Bertrand competition. Trading homogeneous products help to lower the tariff level both in quantity and in price competition.

The remainder of the paper is organized as follows. Section 2 presents the duopoly model of trade and carries out the model analysis of quantity competition and price competition. Section 3 shows the research conclusions.

2. The Model

Suppose that there are two countries, with a firm in each. Both firms produce differentiated products in each country, and the foreign firm exports them to the home country. In other words, the foreign and home firms compete in the domestic market. We denote the foreign and home firms as Firm 1 and Firm 2, respectively. Following Singh & Vives (1984), the utility function of a representative consumer is $U(q_1, q_2) = q_1 + q_2 - \frac{1}{2}(q_1^2 + q_2^2 + 2rq_1q_2)$, where q_i is Firm i's output and $r \in (0, 1]$ measures the product homogeneity (the greater the r, the smaller the product differentiation). The utility function generates the linear demand functions, linear inverse demand functions are given as $q_i = \frac{1 - r - p_i + rp_j}{1 - r^2}$ and $p_i = 1 - q_i - rq_j$ (Chen et al., 2021b), where $i = 1, 2$ and $i \neq j$. Then, consumer surplus is denoted by $CS = \frac{q_1^2 + q_2^2 + 2rq_1q_2}{2}$, where p_i is Firm i's price.

We assume that there is no transportation cost between the two countries. Both the firms have identical technologies with increasing marginal costs. The cost functions of Firm i in each country are assumed to be identical and quadratic, given by $C(q_i) = \frac{1}{2}(q_i^2)$, The home government imposes a tariff on the imports produced by the foreign firm, where the import tariff is denoted by $\lambda \in (0, 1)$. Import tariff revenue is denoted by $R = \lambda q_1$ in the home country. Then, the profits of Firm 1 and Firm 2 can be given as $\pi_1 = p_1 q_1 - \frac{1}{2}(q_1^2) - \lambda q_1$, $\pi_2 = p_2 q_2 - \frac{1}{2}(q_2^2)$.

Domestic social welfare is defined as the sum of a firm's profits, consumer surplus, and import tariff revenues: $sw = \pi_2 + cs + \lambda q_1$. The firms aim to maximize profits, and the home government aims to maximize social welfare by imposing a tariff on imports. The two firms obey either quantity competition or price competition.

The observable two-period delay game is adopted to investigate the endogenous timing issue. In the first stage, the home government chooses the tariff level to maximize domestic social welfare. The second stage includes two periods in which the two firms make strategic decisions endogenously. In the second stage, both firms may choose quantity or price as the competition variable in the first or second periods to maximize their profits under the given tariff. In other words, the two firms declare in which period they choose the competition variable and carry out their choice. Under the decision of competition variable, each firm sets the level of its strategic variable.

In the second stage, there exists three situations in the observable delay game: (i) the simultaneous-move equilibrium, including the Cournot equilibrium and the Bertrand equilibrium; (ii) the Stackelberg equilibrium when the foreign firm is the leader; (iii) the Stackelberg equilibrium when the home firm is the leader. In Situation (i), the firms choose the same strategic variables in the first or second period and engage in Cournot or Bertrand competition. In Situation (ii), the foreign firm chooses the first period, and the home firm chooses the second period. The foreign firm makes the optimal strategy by considering the home firm's reaction. On the contrary, in Situation (iii), the home firm chooses the first period, and the foreign firm chooses the second period. As the first-mover, the home firm can decide its optimal strategy by considering the foreign firm's reaction. In the first stage, the government sets the optimal level of tariff for the foreign firm and conjectures the firms' choice in the second stage. The subgame perfect Nash equilibrium is solved by backward induction.

2.1 Quantity Competition

2.1.1 Cournot competition

In the second stage, we investigate the endogenous timing game in quantity competition. Given the optimal level of tariff, λ, which has already been set in the first stage, the foreign firm and home firm choose their output q_1 and q_2 to maximize the objective functions respectively. Let superscript C denote the Cournot equilibrium. Solving the first-order conditions for both the firms, $\frac{\partial \pi_1}{q_1} = 0$ and $\frac{\partial \pi_2}{q_2} = 0$, we obtain the Cournot equilibrium output as follows:

$$q_1^C = \frac{3(1 - \lambda) - r}{9 - r^2} \tag{1}$$

$$q_2^C = \frac{3 + (\lambda - 1)r}{9 - r^2} \tag{2}$$

The equilibrium variables other than output are shown in Table 1. Normal production requires $q_i^C > 0$, $p_i^C > 0$ and $\pi_i^C > 0$. Then, there is $\lambda \in (0, \widehat{\lambda_0})$, where $\widehat{\lambda_0} = 1 - \frac{r}{3}$.

Table 1. Cournot equilibrium

Variables	Symbols	Equilibrium Results
Firm 1's price	p_1^C	$\frac{2(3 - r) + \lambda(3 - r^2)}{9 - r^2}$
Firm 2's price	p_2^C	$\frac{6 - 2r(1 - \lambda)}{9 - r^2}$
Firm 1's profit	π_1^C	$\frac{3(3\lambda - 3 + r)^2}{2(9 - r^2)^2}$
Firm 2's profit	π_2^C	$\frac{3(3 + (\lambda - 1)r)^2}{2(9 - r^2)}$
Consumer surplus	cs^C	$\frac{2(1 - \lambda)r(r^2 + 3) + (9 - 5r^2)(\lambda^2 - 2\lambda + 2)}{2(9 - r^2)^2}$
Social welfare	sw^C	$\frac{2r^3 + (4\lambda^2 - 2\lambda - 7)r^2 - 6(2 + \lambda)r - 45\lambda^2 + 36\lambda + 45}{2(9 - r^2)^2}$

2.1.2 Stackelberg equilibrium when Firm 1 is the leader

As the first mover in the competition, Firm 1 sets the optimal quantity considering Firm 2's reaction. Firm 1 sets the quantity to maximize its profits by satisfying the first-order condition. Let superscript L denote the Stackelberg equilibrium when Firm 1 is the leader. Then, the equilibrium quantity can be derived as:

$$q_1^L = \frac{3(1 - \lambda) - r}{9 - 2r^2} \tag{3}$$

$$q_2^L = \frac{9 - r^2 - 3r(1 - \lambda)}{3(9 - 2r^2)} \tag{4}$$

Standard calculation gives other equilibrium results shown in Table 2. Normal production requires $q_i^L > 0$, $p_i^L > 0$ and $\pi_i^L > 0$. Then, there is $\lambda \in (0, \widehat{\lambda_0})$.

Table 2. Stackelberg equilibrium when Firm 1 is the leader

Variables	Symbols	Equilibrium Results
Firm 1's price	p_1^L	$\frac{r^3 - 3(1 + \lambda)r^2 + 3(3\lambda - 2r + 6)}{3(9 - 2r^2)}$
Firm 2's price	p_2^L	$\frac{18 - 2r^2 - 6(1 - \lambda)r}{3(9 - 2r^2)}$
Firm 1's profit	π_1^L	$\frac{(3(\lambda - 1) + r)^2}{54 - 12r^2}$
Firm 2's profit	π_2^L	$\frac{(r^2 + 3(1 - \lambda)r - 9)^2}{6(9 - 2r^2)^2}$
Consumer surplus	cs^L	$\frac{7r^4 + 6r(1 - \lambda)(r^2 + 9) - 9(5\lambda^2 - 10\lambda + 12)r^2 + 81(\lambda^2 - 2\lambda + 2)}{18(9 - 2r^2)^2}$
Social welfare	sw^L	$\frac{5r^2 + 6(2 + \lambda)r + 9(5\lambda^2 - 4\lambda - 5)}{18(2r^2 - 9)}$

2.1.3 Stackelberg equilibrium when Firm 2 is the leader

When Firm 2 is the leader in Stackelberg competition, the first-mover advantage is obtained by the home firm, then the optimal output level of Firm 2 will be determined by maximizing its profits with the consideration of Firm 1's reaction. Let superscript F denote the Stackelberg equilibrium when Firm 2 is the leader. Then, the equilibrium quantity can be derived as:

$$q_1^F = \frac{(\lambda - 1)r^2 - 3(r + 3\lambda - 3)}{3(9 - 2r^2)} \tag{5}$$

$$q_2^F = \frac{3 - r(1 - \lambda)}{9 - 2r^2} \tag{6}$$

Standard calculation gives other equilibrium results shown in Table 3.

Table 3. Stackelberg equilibrium when Firm 2 is the leader

Variables	Symbols	Equilibrium Results
Firm 1's price	p_1^F	$\frac{3(3\lambda - 2r + 6) - 2(1 + 2\lambda)r^2}{3(9 - 2r^2)}$
Firm 2's price	p_2^F	$\frac{(6 - r^2)(3 + (\lambda - 1)r)}{3(9 - 2r^2)}$
Firm 1's profit	π_1^F	$\frac{((\lambda - 1)r^2 - 3r - 9\lambda + 9)^2}{6(9 - 2r^2)}$
Firm 2's profit	π_2^F	$\frac{(3 + (\lambda - 1)r)^2}{6(9 - 2r^2)}$
Consumer surplus	cs^F	$\frac{7(1 - \lambda)^2r^4 + 6r(1 - \lambda)(r^2 + 9) - 9(7\lambda^2 - 14\lambda + 12)r^2 + 81(\lambda^2 - 2\lambda + 2)}{18(9 - 2r^2)^2}$
Social welfare	sw^F	$\frac{(10\lambda - 11\lambda^2 + 1)r^4 + 6(7 - \lambda)r^3 + 9(14\lambda^2 - 10\lambda - 15)r^2 - 54(2 + \lambda)r - 81(5\lambda^2 - 4\lambda - 5)}{18(9 - 2r^2)^2}$

2.1.4. Observable delay game

The timing decisions of the market players are endogenously analyzed through the observable delay game. Comparing the objectives of the two firms and the domestic government under three timing scenarios, we find that depending on the value of λ, the Nash equilibrium of the endogenous timing game in quantity competition exists in two cases: (i) If $\lambda \in (0, \widehat{\lambda_2})$, the unique equilibrium is a simultaneous-move Cournot equilibrium. (ii) If $\lambda \in (\widehat{\lambda_2}, \widehat{\lambda_0})$, the unique equilibrium is Stackelberg solution when Firm 2 is the leader. These results are concisely proved as follows. As shown in Figure 1, for all $\lambda \in (0, \widehat{\lambda_0})$ and $r \in (0, 1]$, we get $\pi_1^C < \pi_1^L$ and $\pi_2^F > \pi_2^C > \pi_2^L$. Moreover, when $\pi_1^L \gtreqless \pi_1^F$ if and only if $\lambda \lesseqgtr \widehat{\lambda_1}$. When $\pi_1^C \gtreqless \pi_1^F$ if and only if $\lambda \lesseqgtr \widehat{\lambda_2}$. Because $\widehat{\lambda_2} = \frac{(r-3)(r^3+12r^2-54)}{r^4-36r^2+162} < \widehat{\lambda_1} = \frac{r-3+\sqrt{9-2r^2}}{r} < \widehat{\lambda_0}$, then both (period 1, period 1) and (period 2, period 1) can always be the equilibria.

Figure 1 indicates the region of (r, λ) where an equilibrium result occurs in the observable delay game and what kind of equilibrium it is. Notably, the Stackelberg equilibrium of the observable delay game in which the home firm becomes the leader occurs within a relatively narrow range of the product homogeneity level exogenously given. For example, when $r = 0.6$, the range of tariff level is $\lambda \in (0.7958, 0.8)$. When $r = 1$, the range of tariff level becomes $\lambda \in (0.6457, 0.6667)$.

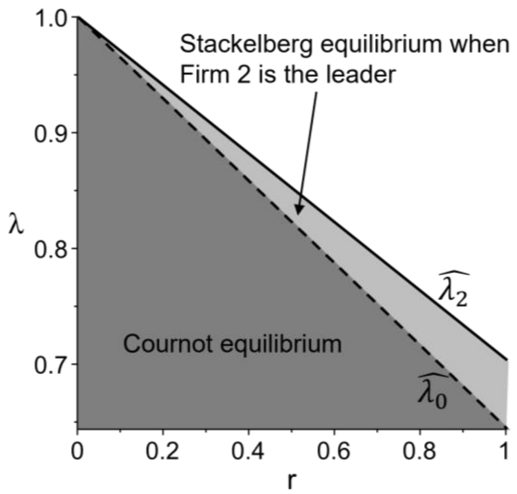

Figure 1. Endogenous timing in quantity competition

2.1.5. The optimal tariff

Given the degree of product homogeneity, r, the government sets the optimal level of tariff on the assumption that it correctly forecasts the possible subsequent result of the endogenous timing game. In the three kinds of scenarios, the first-order conditions for the home government to maximize social welfare are as follows:

$$\frac{\partial sw^C}{\partial \lambda} = \frac{\partial sw^L}{\partial \lambda} = \frac{\partial sw^F}{\partial \lambda} = 0 \tag{7}$$

The optimal tariff levels are given as follows:

$$\lambda^{C*} = \frac{(3-r)(6+r)}{45-4r^2} \tag{8}$$

$$\lambda^{L*} = \frac{6-r}{15} \tag{9}$$

$$\lambda^{F*} = \frac{5r^4-3r^3-45r^2-27r+162}{11r^4-126r^2+405} \tag{10}$$

In order to consider the first stage in which the government decides the optimal tariff level to maximize social welfare, we compare the optimal tariff levels under three kinds of competition modes (λ^{C*}, λ^{L*} and λ^{F*}). First, $\lambda^{F*} < \lambda^{L*} < \lambda^{C*} < \widehat{\lambda_0}$ hold for all $r \in (0.494, 1)$ and $\lambda^{L*} < \lambda^{F*} < \lambda^{C*} < \widehat{\lambda_0}$ hold for all $r \in (0, 0.494)$. In addition, as shown in Figure 2, for each value of $r \in (0, 1]$, there exists $sw^{F*} = sw(\lambda^{F*}) > sw^{L*} = sw(\lambda^{L*}) > sw^{C*} = sw(\lambda^{C*})$. Considering the results of the observable delay game in the second stage, the home government sets the optimal tariff level under Stackelberg competition when the home firm is the leader, as shown in Proposition 1.

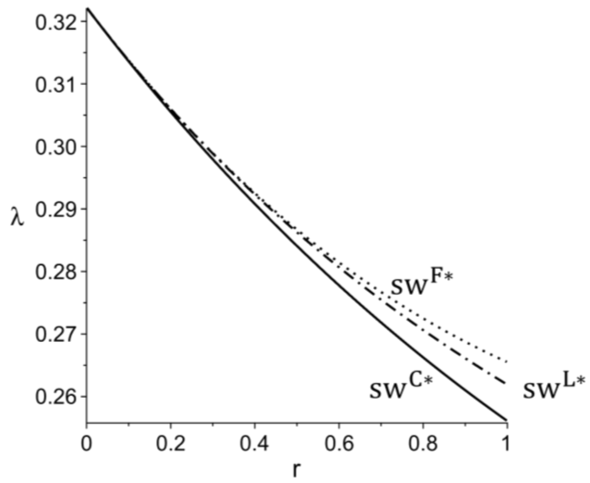

Figure 2. sw^{F*}, sw^{L*}, and sw^{C*}

Proposition 1: The optimal choice for the home government and the two firms is unique, that is the home-leading Stackelberg equilibrium.

As shown in Proposition 1, because the home government has an incentive to decrease λ until $\lambda < \widehat{\lambda_0}$ is achieved (λ^{F*}), each firm and the home government will select the Stackelberg equilibrium when the home firm is the leader in the observable delay game. Considering the goal of the home government, Cournot equilibrium is never realized in the observable delay game. The home government achieves the most favorable outcome with the home-leading Stackelberg competition because $sw^{F*} = sw(\lambda^{F*}) > sw^{L*} = sw(\lambda^{L*}) > sw^{C*} = sw(\lambda^{C*})$ for each value of $r \in (0, 1]$. The home government and the two firms have different intentions in market competition, which indicates that changing the tariff level can realize the objectives of different market participants.

Furthermore, the equilibrium outcomes of home-leading Stackelberg competition are obviously affected by the values of product homogeneity. Corollary 1 can be obtained by analyzing these effects.

Corollary 1: If the two firms engage in home-leading Stackelberg competition, the value of r has the following effects on the equilibrium outcomes (proof see Appendix A).

(i) $\frac{\partial \lambda^{F*}}{\partial r} < 0$.

(ii) If $\lambda \in (0, \lambda_1)$, then $\frac{\partial \pi_1^F}{\partial r} < 0$, if $\lambda \in (\lambda_1, 1)$, then $\frac{\partial \pi_1^F}{\partial r} > 0$.

(iii) If $\lambda \in (0, \lambda_2)$, then $\frac{\partial \pi_2^F}{\partial r} < 0$, if $\lambda \in (\lambda_2, 1)$, then $\frac{\partial \pi_2^F}{\partial r} > 0$.

(iv) $\frac{\partial cs^F}{\partial r} > 0$.

(v) $\frac{\partial sw^F}{\partial r} < 0$.

As shown in Corollary 1, the value of r is negatively correlated with λ^{F*} and sw^F, which means that increasing the product differentiation can enhance the home social welfare and the tariff level. However, the impacts of product differentiation on the home and foreign firms' profits depend on the range of tariff level, which in turn relies on product differentiation. Hence, the tariff level and product differentiation influence how they affect the equilibrium outcomes. We can also find that the greater the product homogeneity, the higher the consumer surplus. Although the product differentiation strategy is essential for international trade, narrowing product differentiation can alleviate the trade frictions caused by steep tariffs in the quantity competition.

2.2 Price Competition

2.2.1 Bertrand competition

We analyze the endogenous timing game in price competition in the second stage. Given the optimal level of tariff, λ, which has already been set by the home government in the first stage, the foreign firm and home firm choose their price p_1 and p_2 to maximize their objective functions respectively. Let superscript B denote the Bertrand equilibrium. Solving the first-order conditions for both the firms, $\frac{\partial \pi_1}{p_1} = 0$ and $\frac{\partial \pi_2}{p_2} = 0$, we obtain the Bertrand equilibrium price as follows:

$$p_1^B = \frac{r^3(r + 1) - (2\lambda + 5)r^2 - 2r + 3\lambda + 6}{r^4 - 7r^2 + 9} \tag{11}$$

$$p_2^B = \frac{(r^2 - 2)(2r^2 - 3)(r^2 + r(1 - \lambda) - 3)}{2r^6 - 17r^4 + 39r^2 - 27} \tag{12}$$

Standard calculation gives the equilibrium results other than the equilibrium price shown in Table 4. Normal production requires $q_i^B > 0$, $p_i^B > 0$, and $\pi_i^B > 0$. Then, there is $\lambda \in (0, \widetilde{\lambda_0})$, where $\widetilde{\lambda_0} = \frac{3 - r^2 - r}{3 - r^2}$.

Table 4. Bertrand equilibrium

Variables	Symbols	Equilibrium Results
Firm 1's quantity	q_1^B	$\frac{(r^2 + r - 3)(r^2 - 1)((\lambda - 1)(r^2 - 3) - r)(r^2 - r - 3)(2r^2 - 3)}{2r^{12} - 33r^{10} + 207r^8 - 629r^6 + 993r^4 - 783r^2 + 243}$
Firm 2's quantity	q_2^B	$\frac{(r^2 + r - 3)(r^2 - 1)(r^2 - (\lambda - 1)r - 3)(r^2 - r - 3)(3 - 2r^2)}{2r^{12} - 33r^{10} + 207r^8 - 629r^6 + 993r^4 - 783r^2 + 243}$
Firm 1's profit	π_1^B	$\frac{(r^2 + r - 3)^2(3 - 2r^2)^3((\lambda - 1)(r^2 - 3) - r)^2(r^2 - r - 3)^2}{2(2r^6 - 17r^4 + 39r^2 - 27)^2(r^4 - 7r^2 + 9)^2}$
Firm 2's profit	π_2^B	$\frac{(r^2 + r - 3)^2(3 - 2r^2)^3(r^2 - (\lambda - 1)r - 3)^2(r^2 - r - 3)^2}{2(2r^6 - 17r^4 + 39r^2 - 27)^2(r^4 - 7r^2 + 9)^2}$
Consumer surplus	cs^B	$\frac{((r^2 - 3)^2 - r^2)^2(r^2 - 1)^2(2r^2 - 3)^2(2r(1 - \lambda)(r^4 - 3r^2 + 3) + (\lambda^2 - 2\lambda + 2)(3r^4 + 11r^2 - 9))}{2(2r^{12} - 33r^{10} + 207r^8 - 629r^6 + 993r^4 - 783r^2 + 243)^2}$
Social welfare	sw^B	$\frac{2(\lambda^2 - \lambda - 1)r^6 - 2r^5 + (19(1 - \lambda^2) + 18\lambda)r^4 + 2(6 + \lambda)r^3 + \left(52\lambda^2 - 11(4\lambda + 5)\right)r^2 - 6(2 + \lambda)r - 9(5\lambda^2 - 4\lambda - 5)}{2(r^2 + r - 3)^2(r^2 - r - 3)^2}$

2.2.2 Stackelberg equilibrium when Firm 1 is the leader

As the first mover in the competition, Firm 1 sets the optimal price with the consideration of Firm 2's reaction. The foreign firm sets the price to maximize profits by satisfying the first-order condition. Let superscript l denote the Stackelberg equilibrium when Firm 1 is the leader. Then, the equilibrium price can be derived as:

$$p_1^l = \frac{(3 + 2\lambda)r^4 + 3r^3 - 3(3\lambda + 5)r^2 - 3(2r - 3\lambda - 6)}{5r^4 - 24r^2 + 27} \tag{13}$$

$$p_2^l = \frac{(2r^2 - 3)(r^2 - 2)\left((\lambda - 1)(r^2 - 3)r + 4r^2 - 9\right)}{10r^6 - 63r^4 + 126r^2 - 81} \tag{14}$$

Standard calculation gives the equilibrium results other than the equilibrium price shown in Table 5. Normal production requires $q_i^l > 0$, $p_i^l > 0$, and $\pi_i^l > 0$. Then, we can obtain $\lambda \in (0, \widetilde{\lambda_0})$.

Table 5. Stackelberg equilibrium when Firm 1 is the leader

Variables	Symbols	Equilibrium Results
Firm 1's quantity	q_1^l	$\frac{(1 - \lambda)(r^2 - 3) + r}{5r^2 - 9}$
Firm 2's quantity	q_2^l	$\frac{(1 - \lambda)(r^2 - 3)r - 4r^2 + 9}{5r^4 - 24r^2 + 27}$
Firm 1's profit	π_1^l	$\frac{((\lambda - 1)(r^2 - 3) - r)^2}{10r^4 - 48r^2 + 54}$
Firm 2's profit	π_2^l	$\frac{(3 - 2r^2)((\lambda - 1)(r^2 - 3)r + 4r^2 - 9)^2}{2(r^2 - 3)^2(5r^2 - 9)^2}$
Consumer surplus	cs^l	$\frac{(1 - \lambda)r(3(1 - \lambda)r^7 - 4r^6 + 28r^4 - 66r^2 + 54) - (29\lambda^2 - 58\lambda + 36)r^6 + 2(51\lambda^2 - 102\lambda + 77)r^4 - 3(51\lambda^2 - 102\lambda + 90)r^2 + 81(\lambda^2 - 2\lambda + 2)}{2(r^2 - 3)^2(5r^2 - 9)^2}$
Social welfare	sw^l	$\frac{(8\lambda - 9\lambda^2 + 1)r^8 + 2(6 - \lambda)r^7 + (94\lambda^2 - 80\lambda - 53)r^6 + 2(\lambda - 40)r^5 + (300\lambda - 366\lambda^2 + 310)r^4 + 6(28 + 5\lambda)r^3 + 9(70\lambda^2 - 56\lambda - 69)r^2 - 54(2 + \lambda)r - 81(5\lambda^2 - 4\lambda + 5)}{2(r^2 - 3)^2(5r^2 - 9)^2}$

2.2.3 Stackelberg equilibrium when Firm 2 is the leader

When Firm 2 obtains the first-mover advantage in Stackelberg competition, the optimal price of the home firm will be determined by maximizing its profits with the consideration of Firm 1's reaction. Let superscript f denote the Stackelberg equilibrium when Firm 2 is the leader. Then, the equilibrium price can be derived as:

$$p_1^f = \frac{(3 - 2r^2)(r^5 - (\lambda + 4)r^4 - 5r^3 + (7\lambda + 17)r^2 + 3(2r - 3\lambda - 6))}{10r^6 - 63r^4 + 126r^2 - 81} \tag{15}$$

$$p_2^f = \frac{3(2 - r^2)(3 + (\lambda - r - 1)r)}{(3 - r^2)(9 - 5r^2)} \tag{16}$$

Standard calculation gives other equilibrium results shown in Table 6. In order to realize normal production ($q_i^f > 0,\ p_i^f > 0,\ \pi_i^f > 0$), there is $\lambda \in (0, \widetilde{\lambda_0})$.

Table 6. Stackelberg equilibrium when Firm 2 is the leader

Variables	Symbols	Equilibrium Results
Firm 1's quantity	q_1^f	$\frac{r^3 + 4(\lambda - 1)r^2 - 3(r + 3\lambda - 3)}{(3 - r^2)(9 - 5r^2)}$
Firm 2's quantity	q_2^f	$\frac{3 + r(\lambda - r - 1)}{9 - 5r^2}$
Firm 1's profit	π_1^f	$\frac{(3 - 2r^2)(r^3 + 4(\lambda - 1)r^2 - 3(r + 3\lambda - 3))^2}{2(3 - r^2)^2(9 - 5r^2)^2}$
Firm 2's profit	π_2^f	$\frac{(r^2 + (1 - \lambda)r - 3)^2}{10r^4 - 48r^2 + 54}$
Consumer surplus	cs^f	$\frac{3r^8 - 2r(1 - \lambda)(r^2(r^4 - 14r^2 - 33) + 27) - (7\lambda^2 - 14\lambda + 36)r^6 + 2(26\lambda^2 - 52\lambda + 77)r^4 - (117\lambda^2 - 234\lambda + 270)r^2 + 81(\lambda^2 - 2\lambda + 2)}{2(3 - r^2)^2(9 - 5r^2)^2}$
Social welfare	sw^f	$\frac{8r^8 + (4\lambda + 6)r^7 + (38\lambda^2 - 36\lambda - 85)r^6 - (50 + 28\lambda)r^5 + (226\lambda - 254\lambda^2 + 346)r^4 + 66(2 + \lambda)r^3 + 9(62\lambda^2 - 52\lambda - 69)r^2 - 54(2 + \lambda)r - 81(5\lambda^2 - 4\lambda - 5)}{2(5r^4 - 24r^2 + 27)^2}$

2.2.4 Observable delay game

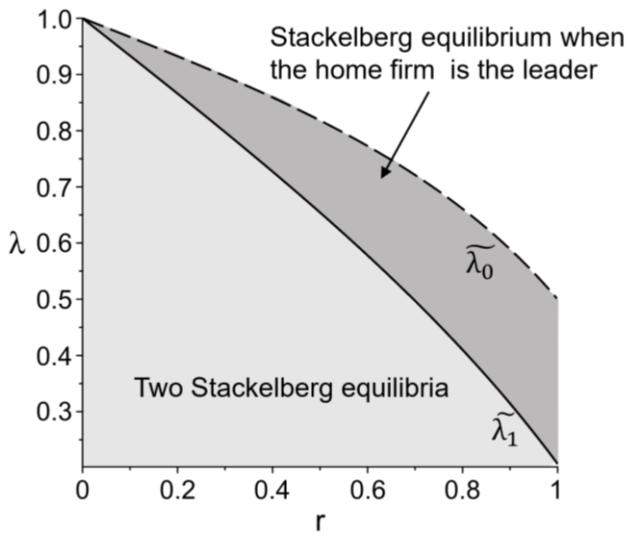

Figure 3. Endogenous timing in price competition

The timing decisions of the two firms and the home government are endogenously explored under observable delay game. Depending on the tariff level, the paper compares the two firms' objective functions in the observable delay game. We find that the Nash equilibrium in price competition includes two cases:

(i) If $\lambda \in (0, \widetilde{\lambda_1})$, there exists two kinds of equilibria and both are the Stackelberg equilibrium as:

$$\widetilde{\lambda_1} = \frac{2r^3 - 3r^2 - \sqrt{63r^4 - 10r^6 - 126r^2 + 81} - 6r + 9}{2r(r^2 - 3)} \in (0,1) \tag{17}$$

(ii) If $\lambda \in (\widetilde{\lambda_1}, \widetilde{\lambda_0})$, the unique equilibrium for the two firms is the Stackelberg one when the home firm is the leader. The results are concisely proved as follows. As shown in Figure 3. For each value of $\lambda \in (0, \widetilde{\lambda_0})$ and $r \in (0,1]$, there are $\pi_1^B < \pi_1^l < \pi_1^f$, $\pi_2^B < \pi_2^l$ and $\pi_2^B < \pi_2^f$. $\pi_2^f \gtrless \pi_2^l$ can be achieved if and only if $\lambda \gtrless \widetilde{\lambda_1} \in (0,1)$. Because $\widetilde{\lambda_1} < \widetilde{\lambda_0}$, if $\lambda \in (0, \widetilde{\lambda_1})$, the timing decisions (period 1, period 2) and (period 2, period 1) are the two kinds of Nash equilibria. If $\lambda \in (\widetilde{\lambda_1}, \widetilde{\lambda_0})$, (period 2, period 1) is the unique equilibrium for foreign and home firms.

Figure 3 illustrates the region of (r, λ) in which the equilibrium happens in the observable delay game and what kind of equilibrium it is. However, unlike the quantity competition, the region where case (ii) holds is limited to a narrow range of the level of tariff. For example, when $r = 0.5$, the range of tariff level is $\lambda \in (0.6543, 0.8182)$. When $r = 1$, the range of tariff level is $\lambda \in (0.2071, 0.5)$. In a word, the region in which case (ii) increases as product homogeneity improves.

2.2.5 The optimal tariff

Standard calculation gives optimal tariff level as:

$$\lambda^{B*} = \frac{(r^4 - r^3 - 5r^2 + r + 6)(r^2 + r - 3)}{2r^6 - 19r^4 + 52r^2 - 45} \tag{18}$$

$$\lambda^{l*} = \frac{4r^6 - r^5 - 28r^4 - 2r^3 + 66r^2 + 9r - 54}{(r^2 - 3)(9r^4 - 40r^2 + 45)} \tag{19}$$

$$\lambda^{f*} = \frac{\big(2(r - 9)r^2 - (14r - 113)\big)r^4 + 3((11r - 78)r^2 - 9(r + 6))}{254r^4 - 38r^6 - 558r^2 + 405} \tag{20}$$

In order to consider the first stage in which the government decides the optimal tariff level to maximize social welfare, we compare the optimal tariff level under three kinds of competition modes (λ^{B*}, λ^{l*} and λ^{f*}). First, irrespective of the value of $r \in (0,1]$, $\lambda^{B*} < \lambda^{f*}$ and $\lambda^{B*} < \lambda^{l*}$ hold for all $\lambda \in (0, \widetilde{\lambda_0})$. In addition, as shown in Figure 4, for each value of $r \in (0,1]$, there exists $sw^{B*} = sw(\lambda^{B*}) > sw^{l*} = sw(\lambda^{l*})$ and $sw^{B*} = sw(\lambda^{B*}) > sw^{f*} = sw(\lambda^{f*})$. Considering the result of the observable delay game in the second stage, the home government sets the optimal tariff level for the foreign firm. Therefore, the Bertrand equilibrium is the optimal choice for the home government of the three equilibria, which is summarized in Proposition 2.

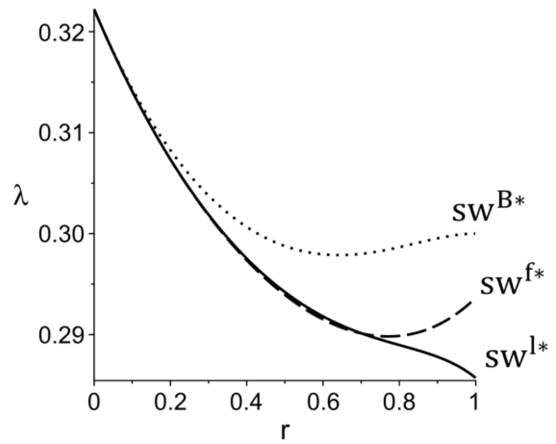

Figure 4. sw^{B*}, sw^{l*}, and sw^{f*}

Proposition 2: Either of the two Stackelberg equilibria is the optimal choice for the two firms in the subsequent observable delay game, which is not in conformity with the decision for maximum-welfare tariff under Bertrand equilibrium.

Proposition 2 implies that the government has an incentive to decrease the level of tariff to λ^{B*}, that is, the case of simultaneous-move game in price can be achieved for welfare maximization. When the home government decides a lower tariff level for the foreign firm, the foreign firm and home firm choose to engage in either of the two Stackelberg price competitions. For the government, the Bertrand price equilibrium is the most favorable result because $sw^{B*} > sw^{l*}$ and $sw^{B*} > sw^{f*}$. However, this equilibrium cannot be realized in the subsequent observable delay game. The foreign firm and the home firm would not take social welfare into the objective function, which is different with the home government. Therefore, the results of observable delay game in multiple equilibria show that different choices and the possibility of coordination failure exist.

Compared with Proposition 1, the contrasting outcomes manifest that choosing different strategic variables gives rise to different equilibria of the timing decision between the foreign and domestic firms. Sequential-move Stackelberg equilibria are chosen in price competition, whereas the Stackelberg equilibrium when the home firm is the leader is chosen in quantity competition. Furthermore, price competition contains multiple alternative Stackelberg equilibria in the observable delay game. Whether the foreign firm would be imposed tariff heavily depends on what kind of equilibrium the government expects. On the contrary, the home government has an incentive to decrease the tariff level in quantity competition, and the unique home-leading Stackelberg equilibrium appears.

Next, Corollary 2 can be obtained by analyzing how the value r affects the equilibrium outcomes.

Corollary 2: If the two firms engage in Bertrand competition, the values of r have the following effects on the equilibrium outcomes when $\lambda \in (0, \widetilde{\lambda_0})$ (Proof See Appendix B).

(i) $\frac{\partial \lambda^{B*}}{\partial r} < 0$.

(ii) $\frac{\partial \pi_1^B}{\partial r} < 0$.

(iii) When $r \in (0, r_1)$, $\lambda \in (0, \widetilde{\lambda_0})$, $\frac{\partial \pi_2^B}{\partial r} < 0$; when $r \in (r_1, 1)$, if $\lambda \in (0, \lambda_3)$, $\frac{\partial \pi_2^B}{\partial r} < 0$; if $\lambda \in (\lambda_3, \widetilde{\lambda_0})$, $\frac{\partial \pi_2^B}{\partial r} > 0$.

(iv) $\frac{\partial cs^B}{\partial r} > 0$.

(v) If $r \in (0, \frac{1}{2})$, $\lambda \in (0, \lambda_4)$, then $\frac{\partial sw^B}{\partial r} > 0$; if $r \in (0, \frac{1}{2})$, $\lambda \in (\lambda_4, \widetilde{\lambda_0})$, then $\frac{\partial sw^B}{\partial r} < 0$.

As shown in Corollary 2, when the two firms engage in Bertrand competition, the value of r is positively correlated with cs^B and negatively correlated with π_1^B, indicating that a high differentiation level leads to a low consumer surplus and high foreign firm's profits. Furthermore, the conclusion of the relationship between r and λ^{B*} agrees with Corollary 1. The value of r has uncertain impacts not only on π_2^B but also on sw^B. Whether the impacts are positive or negative depends on the ranges of r and λ. For example, if $r \in (0, r_1)$ and $\lambda \in (0, \widetilde{\lambda_0})$, then the value of r is negatively correlated with π_2^B, and the home firm should increase the product differentiation. When the values of r and λ are below the critical levels ($\frac{1}{2}$, λ_4), the growth in product differentiation will suppress the home country's social welfare. The above analysis shows that the changes in strategy variables and timing decisions can not affect how product differentiation affects the equilibrium tariffs.

3. Conclusions

This paper constructs a differentiated duopoly competition to investigate the optimal tariff policy involving endogenous timing and the impacts of product differentiation on the equilibrium results under different market structures. The following conclusions are drawn through the analysis.

After exploring the equilibrium of the observable delay game in both quantity competition and price competition, we find that, in quantity competition, the welfare-maximizing home government has an incentive to decrease the tariff level for the foreign firm. The subsequent observable delay game has multiple equilibria, and the firms choose the home-leading Stackelberg equilibrium or Cournot equilibrium. In contrast, the Bertrand equilibrium is the best choice for the home government in price competition. However, the observable delay game has various equilibria, and the firms choose either of the two Stackelberg equilibria. These results indicate that when the home government sets an appropriate tariff imposed on imported products, the possible alternative equilibrium outcomes in the endogenous timing game are limited. If the home government, in an endogenous timing game with quantity competition, decreases the tariff level, the two firms would choose the Stackelberg equilibrium in which the domestic firm is the leader. In price competition, the two firms would choose a Stackelberg equilibrium in which either firm is the leader. Moreover, the home government prefers the simultaneous-move Bertrand equilibrium. The above results show that adjusting the tariff level cannot sufficiently encourage the firms to adopt the welfare-maximizing duopoly because of the different objectives between the home government and firms. Although the resulting equilibrium pattern, under both quantity and price competition, depends on the tariff level and the existing product differentiation, the obtained results in quantity competition are quite different from those obtained in price competition. Therefore, the strategy variables and market structure directly affect the optimal choice in observable delay game. Moreover, increasing product differentiation enhances the home social welfare in home-leading Stackelberg competition but reduces consumer surplus in Bertrand competition. Trading homogeneous products helps to lower the tariff level both in quantity and in price competition.

This study uses a duopoly model capable of producing the optimal tariff policies and timing decisions with two firms in international trade. Therefore, our methodology and main results can be applied to models oriented to analyzing different trade policies on endogenous timing issues. However, future research avenues remain. First, we adopt a linear demand function and a quadratic cost function with only two firms, a foreign firm and a domestic firm, as in many mixed oligopoly studies. Whether the findings can be extended to a more general setting of demand and cost functions needs to be examined in future research. Second, we do not compare the equilibria in quantity and price competition. Because such a comparison requires considering a more complex range of parameters. However, choosing a strategic variable between quantity and price is a matter of great concern for the market players, which should be investigated in the future. Although exploring a more general model is complex, it may yield new insights into this study. In a word, alternative scenarios such as various competition modes with endogenous timing, different numbers of market participants, and more general specifications for the demand and cost functions between firms would be another challenging issue for future research.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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Appendix

Proof of Corollary 1

(i) $\frac{\partial \lambda^{F*}}{\partial r} = \frac{33r^6 - 270r^5 + 1269r^4 + 972r^3 - 7047r^2 + 4374r - 10935}{(11r^4 - 126r^2 + 405)^2} < 0.$

(ii) $\frac{\partial \pi_1^F}{\partial r} = \frac{(2r^2 + 6(\lambda - 1)r + 9)((1-\lambda)(9 - r^2) - 3r)}{(2r^2 - 9)^3}$, $f_1 \triangleq (1-\lambda)(9-r^2) - 3r.$

When $f_1 = 0$, there is $\lambda_1 = \frac{9 - r^2 - 3r}{9 - r^2} \in (0,1)$;

if $\lambda \in (0, \lambda_1)$, then $\frac{\partial \pi_1^F}{\partial r} < 0$;

if $\lambda \in (\lambda_1, 1)$, then $\frac{\partial \pi_1^F}{\partial r} > 0$.

(iii) $\frac{\partial \pi_2^F}{\partial r} = \frac{(r(\lambda - 1) + 3)(3(\lambda - 1) + 2r)}{(2r^2 - 9)^2}$, $f_2 \triangleq 3(\lambda - 1) + 2r.$

When $f_2 = 0$, there is $\lambda_2 = \frac{3 - 2r}{3} \in (0,1)$;

if $\lambda \in (0, \lambda_2)$, then $\frac{\partial \pi_2^F}{\partial r} < 0$;

if $\lambda \in (\lambda_2, 1)$, then $\frac{\partial \pi_2^F}{\partial r} > 0$.

(vi) $\frac{\partial cs^F}{\partial r} = \frac{(\lambda - 1)(2r^4 + 81r^2 + 81) + 3r(10r^2 + 9(3\lambda^2 - 6\lambda + 4))}{3(2r^2 - 9)^3} > 0.$

(v) $\frac{\partial sw^F}{\partial r} = \frac{2(\lambda - 7)r^4 - 6(3\lambda^2 - 14)r^3 + 81(\lambda - 1)r^2 + 27(6\lambda^2 - 6\lambda - 5)r + 81(\lambda + 2)}{3(2r^2 - 9)^3} < 0.$

Proof of Corollary 2

(i) $\frac{\partial \lambda^{B*}}{\partial r} = \frac{-2r^9 + 6r^8 + 32r^7 - 49r^6 - 154r^5 + 119r^4 + 252r^3 - 21r^2 - 108r - 135}{(2r^6 - 19r^4 + 52r^2 - 45)^2} < 0.$

(ii) $\frac{\partial \pi_1^B}{\partial r} = \frac{\left((r^2-3)^2 - r^2\right)^3 (r(\lambda-1)(2r^6 - 10r^4 + 24r^2 - 18) - 27)(2r^2-3)^3((\lambda-1)(r^2-3) - r)}{(2r^6 - 17r^4 + 39r^2 - 27)^3 (r^4 - 7r^2 + 9)^3} < 0.$

(iii) $\frac{\partial \pi_2^B}{\partial r} = \frac{(r(1-\lambda) + r^2 - 3)\left((r^2-3)^2 - r^2\right)^3 (2r^2-3)^3 (2r^7 - 10r^5 + 24r^3 - 18r + (1-\lambda)(4r^6 - 9r^4 - 15r^2 + 27))}{(2r^6 - 17r^4 + 39r^2 - 27)^3 (r^4 - 7r^2 + 9)^3};$

$f_3 \triangleq 2r^7 - 10r^5 + 24r^3 - 18r + (1-\lambda)(4r^6 - 9r^4 - 15r^2 + 27).$

When $f_3 = 0$, there is $\lambda_3 = \frac{(2r^3+3)(r^2+r-3)^2}{r^2(4r^4 - 9r^2 - 15) + 27}$;

when $\lambda_3 - \widetilde{\lambda_0} = 0$, we get $r_1 = \frac{1}{2}\sqrt{6 - 2\sqrt{3}} \in (0,1)$;

when $r \in (0, r_1)$, $\lambda \in (0, \widetilde{\lambda_0})$, $\frac{\partial \pi_2^B}{\partial r} < 0$;

when $r \in (r_1, 1)$, if $\lambda \in (0, \lambda_3)$, $\frac{\partial \pi_2^B}{\partial r} < 0$;

if $\lambda \in (\lambda_3, \widetilde{\lambda_0})$, $\frac{\partial \pi_2^B}{\partial r} > 0$.

(iv) $\frac{\partial cs^B}{\partial r} = \frac{\left((r^2-3)^2 - r^2\right)^3 (2r^2-3)^3 (r^2-1)^3 ((\lambda-1)(3r^8 - 8r^6 - 3r^4 + 18r^2 - 27) - r(\lambda^2 - 2\lambda + 2)(6r^2 - 33r^4 + 59r^2 - 27))}{(2r^{12} - 33r^{10} + 207r^8 - 629r^6 + 993r^4 - 783r^2 + 243)^3} > 0.$

(v) $\frac{\partial sw^B}{\partial r} = \frac{3r^8 - 2(\lambda^2 - \lambda - 1)r^9 + 2(12\lambda^2 - 11\lambda - 12)r^7 - (5\lambda + 23)r^6 - 3(34\lambda^2 - 26\lambda - 37)r^5 + (28\lambda + 39)r^4 + (202\lambda^2 - 128\lambda - 233)r^3 - 36(\lambda-1)r^2 - 27((6\lambda^2 - 4\lambda - 5)r + \lambda + 2)}{((r^2-3)^2 - r^2)^3}.$

When $\frac{\partial sw^B}{\partial r} = 0$,

$\lambda_4 = \frac{(r^2+r-3)(2r^6(r-1) - 14r^5 + 3r(r^3 + 11r^2 - 11) + 4r^2 + 9 + \sqrt{(r^2-r-3)^2(4r(5r^9 + 6r^8 - 48r^7 - 39r^6 + 194r^5 + 71r^4 + 8r^2 - 126) - r^2(1519r^2 - 1262) + 9)})}{4r(r^8 - 12r^6 + 51r^4 - 101r^2 + 81)};$

$f_4 \triangleq (2r^6(r-1) - 14r^5 + 3r(r^3 + 11r^2 - 11) + 4r^2 + 9 + ((r^2 - r - 3)^2(4r(5r^9 + 6r^8 - 48r^7 - 39r^6 + 194r^5 + 71r^4 + 8r^2 - 126) - r^2(1519r^2 - 1262) + 9))^{\frac{1}{2}}.$

When $f_4 = 0$, we get $r_2 = \frac{1}{2}$;

if $r \in (0, \frac{1}{2})$, then $\lambda_5 < 0 < \lambda_4 < \widetilde{\lambda_0}$;

if $\lambda \in (0, \lambda_4)$, then $\frac{\partial sw^B}{\partial r} > 0$;

if $\lambda \in (\lambda_4, \widetilde{\lambda_0})$, then $\frac{\partial sw^B}{\partial r} < 0$.