Journal of Industrial Intelligence
https://www.acadlore.com/journals/JII

Enhancing Multi-Attribute Decision Making with Pythagorean Fuzzy Hamacher Aggregation Operators

Tapas Kumar Paul, Chiranjibe Jana*, Madhumangal Pal

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India

* Correspondence: Chiranjibe Jana (jana.chiranjibe7@gmail.com)

Received: 02-04-2023 **Revised:** 03-06-2023 **Accepted:** 03-18-2023

Citation: T. K. Paul, C. Jana, M. Pal, "Enhancing multi-attribute decision making with Pythagorean fuzzy Hamacher aggregation operators," *J. Ind Intell.*, vol. 1, no. 1, pp. 30–54, 2023. https://doi.org/10.56578/jii010103.

Abstract: The attention of many researchers has been drawn to Pythagorean fuzzy information, which involves Pythagorean fuzzy numbers and their aggregation operators. In this study, the concept of the Pythagorean fuzzy set is discussed, along with the Hamacher t-norm and t-conorm operators. Furthermore, novel aggregation operators are developed using the operational rules of the Hamacher t-norm and t-conorm. The primary objective of this article is to develop a multi-attribute decision-making method in a Pythagorean fuzzy environment using Pythagorean fuzzy Hamacher aggregation operators. It is noted that the Hamacher operator, which is a generalization of the algebraic Einstein operator and contains a parameter, is more potent than some existing operators. Finally, an example of an enterprise application software selection problem is presented to demonstrate the proposed method.

Keywords: Pythagorean fuzzy set; Pythagorean fuzzy number; Hamacher operation; Pythagorean fuzzy Hamacher aggregation operator; Multi-attribute decision making

1 Introduction

The important and efficient role of the MADM problem in various decision-making domains, such as engineering and social science, has been widely recognized. MADM approaches are utilized to process and attribute information to compute a suitable alternative or rank alternatives for decision support. These approaches are exercised in different domains, including engineering technology, operation research, and management science.

Table 1. List of abbreviations

Abbreviation	Full form
AO	Aggregation Operator
DEM	Decision Matrix
DE	Decision Expert
FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
MADM	Multi-Attribute Decision Making
MCDM	Multi-Criteria Decision Making
MF	Membership Function
NMF	Non-Membership Function
OR	Operational Rule
PyFE	Pythagorean Fuzzy Environment
PyFI	Pythagorean Fuzzy Information
PyFN	Pythagorean Fuzzy Number
PyFS	Pythagorean Fuzzy Set

In the Pythagorean Fuzzy Environment (PyFE), various types of traditional decision-making approaches are available. For instance, Liang et al. [1] developed a new extension of the TOPSIS (The Technique for Order of

Preference by Similarity to Ideal Solution) approach with the hesitant PyFE, Zhang [2] proposed decision-making based on similarity measur, Garg [3] proposed strategic decision making with immediate probabilities along with the Pythagorean Fuzzy Numbers (PyFNs), Yu et al. [4] proposed the TOPSIS method in the interval-valued Pythagorean fuzzy environment, Zhang [5] proposed the hierarchical QUALIFLEX (The qualitative flexible multiple criteria) approach in the PyFE, Ren et al. [6] proposed the Pythagorean fuzzy TODIM approach, and Khan et al. [7] proposed the extension of TOPSIS based on the Choquet integral (See Table 1).

The MCDM approach has also been used in some research papers. Fodor and Roubens [8] elaborated on axiomatic concepts and procedures of MCDM in a book, while Greco et al. [9] proposed an MCDM approach that is interpreted in rough set theory. Ho et al. [10] proposed a review article based on supplier evaluation and selection using the MCDM technique, and Kahraman et al. [11] proposed supplier selection using the Analytic Hierarchy Process. Malczewski [12] conducted a GIS based MCDM survey from 1990 to 2004, and Mardani et al. [13] reviewed the literature on studies on energy management problems from 1995 to 2015. Myint et al. [14] proposed the idea of land use and land cover change using the MCDM approach with the help of Markov chain and cellular automata analysis. Pohekar and Ramachandran [15] reviewed works of literature on sustainable energy planning using MCDM, while Rey-Valette et al. [16] proposed an MCDM with a participation-based methodology for selecting sustainable development indicators. Silvestri [17] proposed a multi-criteria risk analysis technique to improve safety in manufacturing systems.

In this present article, the Pythagorean Fuzzy Information (PyFI) is used to solve the MADM problem. Atanassov [18] introduced the notion of Interval Fuzzy Sets (IFS) in 1983, which consist of Membership Function (MF) and Non-Membership Function (NMF) to deal with the uncertainty of an element's belongingness to an FS. The concept of Zadeh's Fuzzy Sets (FS) [19], which was introduced in 1965 and consisted only of MF, was generalized by IFS. Yager [20] introduced the notion of Pythagorean Fuzzy Sets (PyFS), which include more fuzzy information than that of FS and IFS [18]. In other words, PyFS is superior to both FS and IFS in terms of possessing information. For example, while an IFS does not include the fuzzy information $\langle 0.7, 0.5\rangle$ as $0.7 + 0.5 \nleq 1$, it can be included in the PyFS as $0.7^2 + 0.5^2 \leq 1$. It is important to note that a member of an IFS belongs to a PyFS, but the converse may not be valid (see Figure 1).

The Aggregation Operator (AO) is crucial in combining fuzzy information into a single datum and solving a MADM issue. Various research works have been conducted on the MADM approach in the Pythagorean Fuzzy Environment (PyFE) based on Dombi averaging and geometric operators. Jana et al. [21] and Khan et al. [22] proposed Pythagorean fuzzy Dombi AOs. Similarly, Jana et al. [23] introduced Dombi AOs in a bipolar fuzzy environment [24–26], while Rahman et al. [27] developed Pythagorean fuzzy Einstein weighted geometric AOs to solve MCDM problems.

Many research works have utilized Hamacher AOs. Gao [28] proposed Pythagorean fuzzy Hamacher prioritized aggregation operators in MCDM, and Gao et al. [29] introduced dual hesitant bipolar fuzzy Hamacher prioritized aggregation operator in MCDM. Other studies developed intuitionistic fuzzy Hamacher aggregation operators [30], single-valued neutrosophic trapezoidal Hamacher aggregation operators [31], picture fuzzy Hamacher aggregation operators [32], dual hesitant Hamacher aggregation operators [33], Hamacher aggregation operators in the interval-valued intuitionistic fuzzy environment [34], Hamacher aggregation operators using generalized neutrosophic numbers [35], hesitant Pythagorean fuzzy Hamacher aggregation operators [36], hesitant fuzzy Hamacher aggregation operators [37], linguistic intuitionistic fuzzy Hamacher aggregation operators [38], analytical articles regarding the Hamacher AOs in uncertain MCDM problems [39], m-polar fuzzy Hamacher AOs [40], picture fuzzy Hamacher AOs [41], bipolar fuzzy Hamacher AOs [42], and dual hesitant Pythagorean fuzzy Hamacher AOs [43]. Additionally, Wei [44] proposed Hamacher AO in the PyFE, while Wu et al. [45] developed single-valued neutrosophic 2-tuple linguistic Hamacher AOs in MCDM, and Zhou et al. [46] proposed hesitant fuzzy Hamacher AOs in MCDM.

However, the use of Hamacher AOs with the Pythagorean Fuzzy Information (PyFI) is a novel work in the MADM approach, which is discussed in this present article. The authors introduced various Pythagorean fuzzy Hamacher operators, such as Pythagorean fuzzy Hamacher weighted averaging (PyFHWA) operator, Pythagorean fuzzy Hamacher ordered weighted averaging (PyFHOWA) operator, Pythagorean fuzzy Hamacher hybrid averaging (PyFHHA) operator, Pythagorean fuzzy Hamacher weighted geometric (PyFHWG) operator, Pythagorean fuzzy Hamacher ordered weighted geometric (PyFHOWG) operator, and Pythagorean fuzzy Hamacher hybrid geometric (PyFHHG) operator.

In this research article, the PyFI is used in the aggregation process based on Pythagorean fuzzy Hamacher averaging and geometric operators to choose the best alternative of enterprise application software, considering the predefined attributes proposed by the Decision Experts (DEs).

This paper is organized as follows: In section 2, we review some fundamental concepts of Pythagorean Fuzzy Sets (PyFS), including t-norm and t-conorm operators, as well as Pythagorean fuzzy weighted averaging operator, Pythagorean fuzzy ordered weighted averaging operator, Pythagorean fuzzy hybrid averaging operator, Pythagorean fuzzy weighted geometric operator, Pythagorean fuzzy ordered weighted geometric operator, and Pythagorean fuzzy hybrid geometric operator.

In section 3, Hamacher t-norm and t-conorm operators are defined. Section 4 discusses three types of Pythagorean fuzzy averaging operators and their properties, including some theorems. Similarly, section 5 discusses three types of Pythagorean fuzzy geometric operators and their properties, including some theorems.

In section 6, an algorithm is presented for solving a MADM problem based on the Pythagorean fuzzy Hamacher weighted averaging (PyFHWA) operator and Pythagorean fuzzy Hamacher weighted geometric (PyFHWG) operator. A numerical example of selecting the best enterprise application software is provided in the same section. Finally, in section 7, we draw a conclusion.

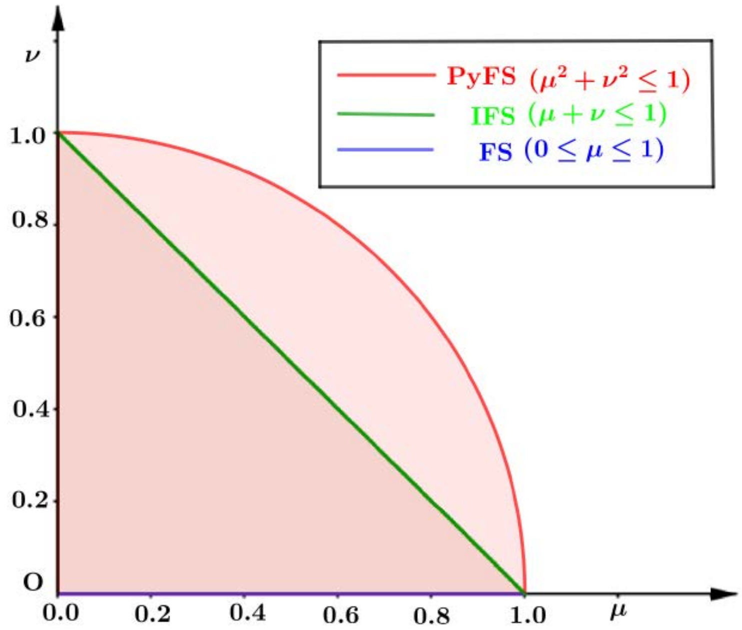

Figure 1. Graphical presentation of PyFS, IFS and FS.

2 Preliminaries

A concise review has been done along with triangular norm (t-norm) and triangular connorm (t-conorm) operators [47, 48].

Definition 2.1 [20, 49, 50] (PyFS)

PyFS on the universe of discourse $\mathfrak{U}$ is defined by $\tilde{\mathfrak{P}} = \left\{\left\langle \varrho, \mathfrak{T}_{\tilde{\mathfrak{P}}}(\varrho), \aleph_{\tilde{\mathfrak{P}}}(\varrho)\right\rangle : \varrho \in \mathfrak{U}\right\}$, where the $MF\mathfrak{T}_{\tilde{P}}$: $\mathfrak{U} \to [0,1], NMF\aleph_{\tilde{P}} : \mathfrak{U} \to [0,1]$ are constrined as $0 \leq \mathfrak{T}^2_{\tilde{\mathfrak{P}}}(\varrho) + \aleph^2_{\tilde{\mathfrak{P}}}(\varrho) \leq 1$. Another function $\pi_{\tilde{P}} : \mathfrak{U} \to [0,1]$ is arisen which is related to the MF and NMF functions by the relation $\mathfrak{T}^2_{\tilde{\mathfrak{P}}}(\varrho) + \aleph^2_{\tilde{\mathfrak{P}}}(\varrho) + \pi^2_{\tilde{\mathfrak{P}}}(\varrho) = 1$ for all $\varrho \in \mathfrak{U}$ i.e., $\pi_{\tilde{\mathfrak{P}}}(\varrho) = \sqrt{1 - \left(\mathfrak{T}^2_{\tilde{\mathfrak{P}}}(\varrho) + \aleph^2_{\tilde{\mathfrak{P}}}(\varrho)\right)}$, which is called the degree of hesitation margin or indeterminacy function. For given $\varrho \in \mathfrak{U}$, $\left\langle \mathfrak{T}_{\tilde{\mathfrak{P}}}(\varrho), \aleph_{\tilde{\mathfrak{P}}}(\varrho)\right\rangle$ is called the Pythagorean fuzzy value corresponding to the PyFS, $\tilde{\mathfrak{P}}$ and it is denoted as $\tilde{\mathfrak{P}} = \left\langle \mathfrak{T}_{\tilde{\mathfrak{P}}}, \aleph_{\tilde{\mathfrak{P}}}\right\rangle$ in short and named as PyFN.

Definition 2.2 [51] (t-norm and t-conorm operators)

t-norm or triangular norm operator is a binary, conjunctive type operator which maps unit square to the unit interval, i.e., t: $[0,1]^2 \to [0,1]$ and satisfies the properties as follows:
(i) $\mathrm{t}(0, \varrho_1) = 0, \mathrm{t}(1, \varrho_1) = \varrho_1$, for all $\varrho_1 \in [0,1]$.
(ii) $\mathrm{t}(\varrho_1, \varrho_2) = \mathrm{t}(\varrho_2, \varrho_1)$, for all $\varrho_1, \varrho_2 \in [0,1]$.
(iii) $\mathrm{t}(\varrho_1, \mathrm{t}(\varrho_2, \varrho_3)) = \mathrm{t}(\mathrm{t}(\varrho_1, \varrho_2), \varrho_3)$, for all $\varrho_1, \varrho_2, \varrho_3 \in [0,1]$.
(iv) $\mathrm{t}(\varrho_1, \varrho_2) \leq \mathrm{t}(\varrho'_1, \varrho'_2)$, for all $\varrho_1 \leq \varrho'_1, \varrho_2 \leq \varrho'_2$ and $\varrho_1, \varrho_2, \varrho'_1, \varrho'_2 \in [0,1]$.

t-conorm or s-norm or triangular conorm operator is a binary, disjunctive type operator which maps unit square to the unit interval, i.e., $\mathrm{s} : [0,1]^2 \to [0,1]$ and satisfies the following properties as follows:
(i) $\mathrm{s}(0, \varrho_1) = \varrho_1, \mathrm{s}(1, \varrho_1) = 1$.
(ii) $\mathrm{s}(\varrho_1, \varrho_2) = \mathrm{s}(\varrho_2, \varrho_1)$, for all $\varrho_1, \varrho_2 \in [0,1]$.
(iii) $\mathrm{s}(\varrho_1, \mathrm{s}(\varrho_2, \varrho_3)) = \mathrm{s}(\mathrm{s}(\varrho_1, \varrho_2), \varrho_3)$, for all $\varrho_1, \varrho_2, \varrho_3 \in [0,1]$.
(iv) $\mathrm{s}(\varrho_1, \varrho_2) \leq \mathrm{s}(\varrho'_1, \varrho'_2)$, for all $\varrho_1 \leq \varrho'_1, \varrho_2 \leq \varrho'_2$ and for all $\varrho_1, \varrho_2, \varrho'_1, \varrho'_2 \in [0,1]$.

Both the operators are related by the relation $\mathrm{s}(a, b) = 1 - \mathrm{t}(1 - a, 1 - b)$ i.e., they satisfy the De'Morgan's duality for all $(a, b) \in [0, 1]^2$.

Definition 2.3 [52] (Score function): Let PyFS($\mathfrak{U}$) denotes all PyFSs on the universe of discourse $\mathfrak{U}$. The score function $\wp = \langle \mathfrak{T}_{\tilde{\wp}}, \aleph_{\tilde{\wp}} \rangle \in$ PyFS($\mathfrak{U}$) is denoted as $Sc(\tilde{\wp})$ and defined as $Sc(\tilde{\wp}) = \mathfrak{T}^2_{\tilde{\wp}} - \aleph^2_{\tilde{\wp}}$. Clearly, $Sc(\tilde{\wp}) \in [-1, 1]$. For any two PyFNs $\tilde{\wp_1}$ and $\tilde{\wp_2}$, if $Sc(\tilde{\wp_1}) > Sc(\tilde{\wp_2})$ then $\tilde{\wp_1} > \tilde{\wp_2}$ and if $Sc(\tilde{\wp_1}) = Sc(\tilde{\wp_2})$ then $\tilde{\wp_1} = \tilde{\wp_2}$.

Sometimes, the score function in definition-2.3 may give an unreasonable result. For example, the score values of two PyFNs, $\langle 0.5, 0.5 \rangle$ and $\langle 0.6, 0.6 \rangle$, remain the same and which implies that the two PyFNs are equal. However, clearly, it is seen that they never are equal. It was pointed out by Peng and Yang [53] and defined the accuracy function of PyFNs.

Definition 2.4 [53] (Accuracy function): Let PyFS($\mathfrak{U}$) denotes all PyFSs on the universe of discourse $\mathfrak{U}$. The accuracy function of $\tilde{\wp} = \langle \mathfrak{T}_{\tilde{\wp}}, \aleph_{\tilde{\wp}} \rangle \in PyFS(\mathfrak{U})$ is denoted as $Ac(\tilde{\wp})$ and defined as $Ac(\tilde{\wp}) = \mathfrak{T}^2_{\tilde{\wp}} + \aleph^2_{\tilde{\wp}}$. Clearly, $Ac(\tilde{\wp}) \in [0, 1]$.

The score function and accuracy function are useful tools to determine the order of a given set of PyFNs. Hence PyFS($\mathfrak{U}$), with the score function and accuracy function, forms a totally ordered set, and the order of two PyFNs are proposed by Peng and Yang [53] as follows:
(i) If $Sc(\tilde{\wp_1}) < Sc(\tilde{\wp_2})$, then $\tilde{\wp_1} < \tilde{\wp_2}$.
(ii) If $Sc(\tilde{\wp_1}) > Sc(\tilde{\wp_2})$, then $\tilde{\wp_1} > \tilde{\wp_2}$,
(iii) If $Sc(\tilde{\rho_1}) = Sc(\tilde{\wp_2})$, then
(a) If $Ac(\tilde{\wp_1}) < Ac(\tilde{\wp_2})$, then $\tilde{\wp_1} < \tilde{\wp_2}$,
(b) If $Ac(\tilde{\wp_1}) > Ac(\tilde{\wp_2})$, then $\tilde{\wp_1} > \tilde{\wp_2}$,
(c) If $Ac(\tilde{\wp_1}) = Ac(\tilde{\wp_2})$, then $\tilde{\wp_1} \simeq \tilde{\wp_2}$,
where $\tilde{\wp_1} = \left\langle \tilde{T}_{\tilde{\wp_1}}, \aleph_{\wp_1} \right\rangle \in PyFS(\mathfrak{U})$ and $\tilde{\wp_2} = \left\langle \tilde{T}_{\tilde{\wp_2}}, \aleph_{\tilde{\wp_2}} \right\rangle \in PyFS(\mathfrak{U})$.

Definition 2.5 [54–56] (Lattice structure of PyFNs):
Let PyFN($\mathfrak{U}$) be the set of Pythagorean fuzzy numbers on $\mathfrak{U}$ and $\leq_L$ be a partial order relation defined on PyFN($\mathfrak{U}$). $\tilde{\wp_1} = \langle \mathfrak{T}_{\tilde{\wp_1}}, \aleph_{\tilde{\wp_1}} \rangle$, $\tilde{\wp_2} = \langle \mathfrak{T}_{\tilde{\wp_2}}, \aleph_{\tilde{\wp_2}} \rangle$ be two Pythagorean fuzzy numbers on PyFN($\mathfrak{U}$). Now $\tilde{\wp_1} \leq_L \tilde{\wp_2} \implies \mathfrak{T}_{\tilde{\wp_1}} \leq \mathfrak{T}_{\tilde{\wp_2}}$ and $\aleph_{\tilde{\wp_1}} \geq \aleph_{\tilde{\wp_2}}$. Thus $(PyFN(\mathfrak{U}), \leq_L)$ forms a lattice with the partial order relation defined above containing $\langle 0, 1 \rangle$ as bottom element and $\langle 1, 0 \rangle$ as top element of the Lattice.

Lemma 2.1 Let $\tilde{a} = \langle a_1, a_2 \rangle$, $\tilde{b} = \langle b_1, b_2 \rangle$ be two PyFNs. If $\tilde{a} \leq_L \tilde{b}$ then $\tilde{a} \leq \tilde{b}$ but the converse may not be true.
proof: Here, $\tilde{a} = \langle a_1, a_2 \rangle$, $\tilde{b} = \langle b_1, b_2 \rangle$. We know $\tilde{a} \leq_L \tilde{b}$ implies $a_1 \leq b_1$ and $a_2 \geq b_2$.
Now, $Sc(\tilde{a}) = a_1^2 - a_2^2 \leq b_1^2 - b_2^2 = Sc(\tilde{b})$ i.e., $Sc(\tilde{a}) \leq Sc(\tilde{b})$.
Case-1: If $Sc(\tilde{a}) < Sc(\tilde{b})$, then $\tilde{a} < \tilde{b}$.

Case-2: If $Sc(\tilde{a}) = Sc(\tilde{b})$, then we have to check corresponding accuracy values.
Since, $a_1 \leq b_1$ and $a_2 \geq b_2$ we suppose that $b_1 = a_1 + p$ and $a_2 = b_2 + q$, where the scalars $p, q \geq 0$.
Then $\tilde{a} = \langle a_1, b_2 + q \rangle$ and $\tilde{b} = \langle a_1 + p, b_2 \rangle$.
Therefore $Sc(\tilde{a}) = Sc(\tilde{b})$ implies that $a_1^2 - (b_2 + q)^2 = (a_1 + p)^2 - b_2^2$.
$i.e.$, $2pa_1 + 2qb_2 + p^2 + q^2 = 0$, which is possible for any a_1, b_2 only when $p = 0$ and $q = 0$ simultaneously.
Then, $Ac(\tilde{a}) = a_1^2 + (b_2 + q)^2 = a_1^2 + b_2^2$, as $q = 0$.
$Ac(\tilde{b}) = (a_1 + p)^2 + b_2^2 = a_1^2 + b_2^2$, as $p = 0$.
Thus, in this case $Sc(\tilde{a}) = Sc(\tilde{b})$ and $Ac(\tilde{a}) = Ac(\tilde{b})$.
Hence, $\tilde{a} = \tilde{b}$.
Hence, from Case-1 and Case-2 we can write that if $\tilde{a} \leq_L \tilde{b}$ then $\tilde{a} \leq \tilde{b}$.

To prove the converse part, we take $\tilde{a} = \langle 0.7, 0.6 \rangle$ and $\tilde{b} = \langle 0.5, 0.2 \rangle$.
Now $Sc(\tilde{a}) = 0.49 - 0.36 = 0.13$ and $Sc(\tilde{b}) = 0.25 - 0.04 = 0.21$.
Thus, $\tilde{a} \leq \tilde{b}$. But it does not imply $\tilde{a} \leq_L \tilde{b}$ because, although it satisfies $a_2 \geq b_2$, it does not satisfy the condition $a_1 \leq b_1$.

Properties 1 [53] For any two PyFNs, $\tilde{\wp_1} = \langle \mathfrak{T}\tilde{\wp_1}, \aleph_{\tilde{\wp_1}} \rangle$,$\tilde{\wp_2} = \langle \mathfrak{T}_{\tilde{\wp_2}}, \aleph_{\tilde{\wp_2}} \rangle$ defined on the universe of discourse $\mathfrak{U}$, the containment, equality, union, intersection and complement operational laws respectively are as follows:
(i) $\tilde{\wp_1} \subseteq \tilde{\wp_2}$ iff $\mathfrak{T}_{\tilde{\wp_1}}(\varrho) \leq \mathfrak{T}_{\tilde{\wp_2}}(\varrho)$,$\aleph_{\tilde{\wp_1}}(\varrho) \geq \aleph_{\tilde{\wp_2}}(\varrho)$, for all $\varrho \in \mathfrak{U}$.
(ii) $\tilde{\wp_1} = \tilde{\wp_2}$ iff $\tilde{\wp_1} \subseteq \tilde{\wp_2}$ and $\tilde{\wp_1} \supseteq \tilde{\wp_2}$.
(iii) $\tilde{\wp_1} \cup \tilde{\wp_2} = \langle max\{\mathfrak{T}_{\tilde{\wp_1}}, \mathfrak{T}_{\tilde{\wp_2}}\}, min\{\aleph_{\tilde{\wp_1}}, \aleph_{\tilde{\wp_2}}\} \rangle$.
(iv) $\tilde{\wp_1} \cap \tilde{\wp_2} = \langle min\{\mathfrak{T}_{\tilde{\wp_1}}, \mathfrak{T}_{\tilde{\wp_2}}\}, max\{\aleph_{\tilde{\wp_1}}, \aleph_{\tilde{\wp_2}}\} \rangle$.
(v) $\tilde{\wp_1}^c = \langle \aleph_{\tilde{\wp_1}}, \mathfrak{T}_{\tilde{\wp_1}} \rangle$.

Definition 2.6 [49, 52] (Operations on PyFNs):
For any three PyFNs, $\tilde{\wp} = \langle \mathfrak{T}_{\tilde{\wp}}, \aleph_{\tilde{\wp}} \rangle$,$\tilde{\wp_1} = \langle \mathfrak{T}_{\tilde{\wp_1}}, \aleph_{\tilde{\wp_1}} \rangle$,$\tilde{\wp_2} = \langle \mathfrak{T}_{\tilde{\wp_2}}, \aleph_{\tilde{\wp_2}} \rangle$ in PyNS($\mathfrak{U}$) and for scalar $\tau > 0$ the basic operational rules on PyFNs are as follows:

(i) $\tilde{\wp_1} \oplus \tilde{\wp_2} = \langle \sqrt{\mathfrak{T}^2_{\tilde{\wp_1}} + \mathfrak{T}^2_{\tilde{\wp_2}} - \mathfrak{T}^2_{\tilde{\wp_1}} \mathfrak{T}^2_{\tilde{\wp_2}}}, \aleph_{\tilde{\wp_1}} \aleph_{\tilde{\wp_2}} \rangle$.

(ii) $\tilde{\wp_1} \otimes \tilde{\wp_2} = \langle \mathfrak{T}_{\tilde{\wp_1}} \mathfrak{T}_{\tilde{\wp_2}}, \sqrt{\aleph^2_{\tilde{\wp_1}} + \aleph^2_{\tilde{\wp_2}} - \aleph^2_{\tilde{\wp_1}} \aleph^2_{\tilde{\wp_2}}} \rangle$.

(iii) $\tau \tilde{\wp} = \langle \sqrt{1 - (1 - \mathfrak{T}^2_{\tilde{\wp}})^\tau}, \aleph^\tau_{\tilde{\wp}} \rangle$.

(iv) $\tilde{\wp}^\tau = \langle \mathfrak{T}^\tau_{\tilde{\wp}}, \sqrt{1 - (1 - \aleph^2_{\tilde{\wp}})^\tau} \rangle$.

Yager [57] introduced some properties on the operational laws of PyFNs, which are given below:

Theorem 2.1 For any three PyFNs, $\tilde{\wp_1} = \langle \mathfrak{T}_{\tilde{\wp_1}}, \aleph_{\tilde{\wp_1}} \rangle$, $\tilde{\wp_2} = \langle \mathfrak{T}_{\tilde{\wp_2}}, \aleph_{\tilde{\wp_2}} \rangle$, $\tilde{\wp_3} = \langle \mathfrak{T}_{\tilde{\wp_3}}, \aleph_{\tilde{\wp_3}} \rangle$ in PyFN($\mathfrak{U}$) and for any scalar $\tau_1 > 0$, $\tau_2 > 0$.

(i) $\tilde{\wp_1} \oplus \tilde{\wp_2} = \tilde{\wp_2} \oplus \tilde{\wp_1}$.

(ii) $\tilde{\wp_1} \otimes \tilde{\wp_2} = \tilde{\wp_2} \otimes \tilde{\wp_1}$.

(iii) $\tau_1(\tilde{\wp_1} \oplus \tilde{\wp_2}) = \tau_1\tilde{\wp_1} \oplus \tau_1\tilde{\wp_2}$.

(iv) $(\tilde{\wp_1} \otimes \tilde{\wp_2})^{\tau_1} = \tilde{\wp_1}^{\tau_1} \otimes \tilde{\wp_2}^{\tau_1}$.

(v) $\tau_1\tilde{\wp_1} \oplus \tau_2\tilde{\wp_1} = (\tau_1 + \tau_2)\tilde{\wp_1}$.

(vi) $\tilde{\wp_1}^{\tau_1} \otimes \tilde{\wp_1}^{\tau_2} = \tilde{\wp_1}^{\tau_1+\tau_2}$.

(vii) $\tilde{\wp_1} \oplus (\tilde{\wp_2} \oplus \tilde{\wp_3}) = (\tilde{\wp_1} \oplus \tilde{\wp_2}) \oplus \tilde{\wp_3}$.

(viii) $\tilde{\wp_1} \otimes (\tilde{\wp_2} \otimes \tilde{\wp_3}) = (\tilde{\wp_1} \otimes \tilde{\wp_2}) \otimes \tilde{\wp_3}$.

To aggregate a given set of PyFNs, some Pythagorean averaging and geometric type AOs are used. The basic Pythagorean fuzzy AOs which are constructed on the basis of binary operators $\oplus$, $\otimes$ defined earlier on PyFNs($\mathfrak{U}$) are defined below.

Definition 2.7 [58, 59] (Pythagorean fuzzy weighted averaging (PyFWA) operator)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFWA operator is a mapping $PyFWA_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFWA_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \bigoplus_{\jmath=1}^{\hbar} (\ell_\jmath \tilde{\wp_\jmath}),$$

where, $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ be a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$.

Hence, $PyFWA_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus ... \oplus \ell_\hbar\tilde{\wp_\hbar} = \left\langle \sqrt{1 - \prod_{\jmath=1}^{\hbar}(1 - \mathfrak{T}^2_{\tilde{\wp_\jmath}})^{\ell_\jmath}}, \prod_{\jmath=1}^{\hbar} \aleph^{\ell_\jmath}_{\tilde{\wp_\jmath}} \right\rangle$.

Definition 2.8 [60] (Pythagorean fuzzy ordered weighted averaging (PyFOWA) operator)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFOWA operator is a mapping $PyFOWA_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFOWA_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \bigoplus_{\jmath=1}^{\hbar} (\ell_\jmath \tilde{\wp}_{\sigma(\jmath)}),$$

where, $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of (1,2,...,$\hbar$) such that $\tilde{\wp}_{\sigma(\jmath-1)} \geq \tilde{\wp}_{\sigma(\jmath)}$ for all $\jmath = 2, 3, ..., \hbar$.

Hence, $PyFOWA_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus ... \oplus \ell_\hbar\tilde{\wp_\hbar} = \left\langle \sqrt{1 - \prod_{\jmath=1}^{\hbar}(1 - \mathfrak{T}^2_{\tilde{\wp}_{\sigma(\jmath)}})^{\ell_\jmath}}, \prod_{\jmath=1}^{\hbar} \aleph^{\ell_\jmath}_{\tilde{\wp}_{\sigma(\jmath)}} \right\rangle$.

Definition 2.9 [58, 61] (Pythagorean fuzzy hybrid averaging (PyFHA) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHA operator is a mapping $PyFHA_{\ell,\Omega} : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \bigoplus_{\jmath=1}^{\hbar} (\ell_\jmath \tilde{\wp}^*_{\sigma(\jmath)}),$$

where, $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$ and $\tilde{\wp}^*_\jmath = \hbar \Omega_\jmath \tilde{\wp}_\jmath$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of $(1, 2, ..., \hbar)$ such that $\tilde{\wp}^*_{\sigma(\jmath-1)} \geq \tilde{\wp}^*_{\sigma(\jmath)}$ for all $\jmath = 2, 3, ..., \hbar$ and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\hbar)^T$ is a associated weight vector such that $\Omega_\jmath \in [0, 1]$ for all $\jmath = 1, 2, 3, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \Omega_\jmath = 1$.

Hence, $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \ell_1 \tilde{\wp}^*_{\sigma(1)} \oplus \ell_2 \tilde{\wp}^*_{\sigma(2)} \oplus ... \oplus \ell_\hbar \tilde{\wp}^*_{\sigma(\hbar)} = \left\langle \sqrt{1 - \prod_{\jmath=1}^{\hbar} (1 - \mathfrak{T}^2_{\tilde{\wp}^*_{\sigma(\jmath)}})^{\ell_\jmath}}, \prod_{\jmath=1}^{\hbar} \aleph^{\ell_\jmath}_{\tilde{\wp}^*_{\sigma(\jmath)}} \right\rangle$.

Lemma 2.2 If $\ell = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ then $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = PyFWA_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

proof: It is given that $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ be an associated weight vector.
Now we have $\tilde{\wp}^*_\jmath = 4\Omega_\jmath \tilde{\wp}_\jmath$ for $\jmath = 1, 2, 3, 4$.
Suppose, without loss of generality that, $\tilde{\wp}^*_2 \geq \tilde{\wp}^*_1 \geq \tilde{\wp}^*_4 \geq \tilde{\wp}^*_3$ i.e., $\tilde{\wp}^*_{\sigma(1)} \geq \tilde{\wp}^*_{\sigma(2)} \geq \tilde{\wp}^*_{\sigma(3)} \geq \tilde{\wp}^*_{\sigma(4)}$.
Hence, $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = \bigoplus_{\jmath=1}^{4} (\ell_\jmath \tilde{\wp}^*_{\sigma(\jmath)})$
$= \ell_1 \tilde{\wp}^*_{\sigma(1)} \oplus \ell_2 \tilde{\wp}^*_{\sigma(2)} \oplus \ell_3 \tilde{\wp}^*_{\sigma(3)} \oplus \ell_4 \tilde{\wp}^*_{\sigma(4)}$
$= \frac{1}{4}[\tilde{\wp}^*_{\sigma(1)} \oplus \tilde{\wp}^*_{\sigma(2)} \oplus \tilde{\wp}^*_{\sigma(3)} \oplus \tilde{\wp}^*_{\sigma(4)}] = \frac{1}{4}[\tilde{\wp}^*_2 \oplus \tilde{\wp}^*_1 \oplus \tilde{\wp}^*_4 \oplus \tilde{\wp}^*_3] = \frac{1}{4}[4\Omega_2 \tilde{\wp}_2 \oplus 4\Omega_1 \tilde{\wp}_1 \oplus 4\Omega_4 \tilde{\wp}_4 \oplus 4\Omega_3 \tilde{\wp}_3] = \bigoplus_{\jmath=1}^{4} (\Omega_\jmath \tilde{\wp}_\jmath) = PyFWA_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

Theorem 2.2 If $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T = (\frac{1}{\hbar}, \frac{1}{\hbar}, ..., \frac{1}{\hbar})^T$ then $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = PyFWA_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$.

proof: The proof is similar to the proof of Theorem 4.11.

Lemma 2.3

If $\Omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ then $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = PyFOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

proof: Now we have $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)^T$ as weight vectors.
Now,we have $\tilde{\wp}^*_\jmath = 4\Omega_\jmath \tilde{\wp}_\jmath$, which becomes $\tilde{\wp}^*_\jmath = \tilde{\wp}_\jmath \; \forall \, \jmath = 1, 2, 3, 4$.
Suppose, without loss of generality, $\tilde{\wp}^*_2 \geq \tilde{\wp}^*_1 \geq \tilde{\wp}^*_4 \geq \tilde{\wp}^*_3$ i.e.,
$\tilde{\wp}^*_{\sigma(1)} \geq \tilde{\wp}^*_{\sigma(2)} \geq \tilde{\wp}^*_{\sigma(3)} \geq \tilde{\wp}^*_{\sigma(4)}$

$PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = \bigoplus_{\jmath=1}^{4} (\ell_\jmath \tilde{\wp}^*_{\sigma(\jmath)}) = \ell_1 \tilde{\wp}^*_{\sigma(1)} \oplus \ell_2 \tilde{\wp}^*_{\sigma(2)} \oplus \ell_3 \tilde{\wp}^*_{\sigma(3)} \oplus \ell_4 \tilde{\wp}^*_{\sigma(4)}$

$= \bigoplus_{\jmath=1}^{4} (\ell_\jmath \tilde{\wp}_{\sigma(\jmath)}) = PyFOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

Theorem 2.3 If $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\hbar)^T = (\frac{1}{\hbar}, \frac{1}{\hbar}, ..., \frac{1}{\hbar})^T$ then $PyFHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$ $= PyFOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$.

proof: The proof is similar to the proof of Theorem 4.12.
It is clear from Theorem 2.2, and Theorem 2.3 that PyFWA, PyFOWA operators are the particular cases of PyFHA operator or PyFHA operator is the generalization of PyFWA and PyFOWA operators.

Definition 2.10 [27, 41] (Pythagorean fuzzy weighted geometric (PyFWG) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of Pythagorean fuzzy numbers in PyFN($\mathfrak{U}$). PyFWG operator is a mapping $PyFWG_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFWG_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \bigotimes_{\jmath=1}^{\hbar} (\tilde{\wp}_\jmath)^{\ell_\jmath},$$

where $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$.

That is $PyFWG_{\ell}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \tilde{\wp}_1{}^{\ell_1} \otimes \tilde{\wp}_2{}^{\ell_2} \otimes ... \otimes \tilde{\wp}_\hbar{}^{\ell_\hbar} = \left\langle \prod_{\jmath=1}^{\hbar} \mathfrak{T}_{\tilde{\wp}_\jmath}^{\ell_\jmath}, \sqrt{1 - \prod_{\jmath=1}^{\hbar}(1 - \aleph_{\tilde{\wp}_\jmath}^2)^{\ell_\jmath}} \right\rangle.$

Definition 2.11 [62] (Pythagorean fuzzy ordered weighted geometric (PyFOWG) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFOWG operator is a mapping $PyFOWG_\ell : \mathfrak{P}^\hbar \rightarrow \mathfrak{P}$ which is defined below.

$$PyFOWG_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \bigotimes_{\jmath=1}^{\hbar} (\tilde{\wp}_{\sigma(\jmath)}^{\ell_\jmath}),$$

where, $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of (1,2,...,$\hbar$) such that $\tilde{\wp}_{\sigma(\jmath-1)} \geq \tilde{\wp}_{\sigma(\jmath)}$ for all $\jmath = 2, 3, ..., \hbar$.
That is $PyFOWG_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$

$$= \tilde{\wp}_{\sigma(1)}^{\ell_1} \otimes \tilde{\wp}_{\sigma(2)}^{\ell_2} \otimes ... \otimes \tilde{\wp}_{\sigma(\hbar)}^{\ell_\hbar} = \left\langle \prod_{\jmath=1}^{\hbar} \mathfrak{T}_{\tilde{\wp}_{\sigma(\jmath)}}^{\ell_\jmath}, \sqrt{1 - \prod_{\jmath=1}^{\hbar}(1 - \aleph_{\tilde{\wp}_{\sigma(\jmath)}}^2)^{\ell_\jmath}} \right\rangle.$$

Definition 2.12 [63] (Pythagorean fuzzy hybrid geometric (PyFHG) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHG operator is a mapping $PyFHG_{\ell,\Omega} : \mathfrak{P}^\hbar \rightarrow \mathfrak{P}$ which is defined below.

$$PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \bigotimes_{\jmath=1}^{\hbar} (\tilde{\wp}_{\sigma(\jmath)}^*)^{\ell_\jmath},$$

where, $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$ and $\tilde{\wp}_\jmath^* = (\tilde{\wp}_\jmath)^{\hbar\Omega_\jmath}$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of $(1, 2, ..., \hbar)$ such that $\tilde{\wp}_{\sigma(\jmath-1)}^* \geq \tilde{\wp}_{\sigma(\jmath)}^*$ for all $\jmath = 2, 3, ..., \hbar$ and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\hbar)^T$ is a associated weight vector such that $\Omega_\jmath \in [0, 1]$ for all $\jmath = 1, 2, 3, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \Omega_\jmath = 1$.

That is $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$

$$= (\tilde{\wp}_{\sigma(1)}^*)^{\ell_1} \otimes (\tilde{\wp}_{\sigma(2)}^*)^{\ell_2} \otimes ... \otimes (\tilde{\wp}_{\sigma(\hbar)}^*)^{\ell_\hbar} = \left\langle \prod_{\jmath=1}^{\hbar} \mathfrak{T}_{\tilde{\wp}_{\sigma(\jmath)}^*}^{\ell_\jmath}, \sqrt{1 - \prod_{\jmath=1}^{\hbar}(1 - \aleph_{\tilde{\wp}_{\sigma(\jmath)}^*}^2)^{\ell_\jmath}} \right\rangle.$$

Lemma 2.4 If $\ell = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ then $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = PyFWG_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

Proof: We have $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ as weight vectors.
Now, $\tilde{\wp}_\jmath^* = (\tilde{\wp}_\jmath)^{4\Omega_\jmath}$ for $\jmath = 1, 2, 3, 4$.
Suppose, without loss of generality, that $\tilde{\wp}_2^* \geq \tilde{\wp}_1^* \geq \tilde{\wp}_4^* \geq \tilde{\wp}_3^*$
i.e., $\tilde{\wp}_{\sigma(1)}^* \geq \tilde{\wp}_{\sigma(2)}^* \geq \tilde{\wp}_{\sigma(3)}^* \geq \tilde{\wp}_{\sigma(4)}^*$

Hence, $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = \bigotimes_{\jmath=1}^{4} (\tilde{\wp}_{\sigma(\jmath)}^*)^{\ell_\jmath}$

$$= (\tilde{\wp}_{\sigma(1)}^*)^{\ell_1} \otimes (\tilde{\wp}_{\sigma(2)}^*)^{\ell_2} \otimes (\tilde{\wp}_{\sigma(3)}^*)^{\ell_3} \otimes (\tilde{\wp}_{\sigma(4)}^*)^{\ell_4}$$

$$= \bigotimes_{\jmath=1}^{4} (\Omega_\jmath \tilde{\wp}_\jmath) = PyFWG_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4).$$

Theorem 2.4 If $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T = (\frac{1}{\hbar}, \frac{1}{\hbar}, ..., \frac{1}{\hbar})^T$ then $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = PyFWG_\Omega(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$.

proof: The proof is similar as Theorem 5.11.

Lemma 2.5 If $\Omega = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ then $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = PyFOWG_\ell(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4)$.

Proof: We have $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)^T$ as weight vectors.
Now, $\tilde{\wp}_\jmath^* = (\tilde{\wp}_\jmath)^{4\Omega_\jmath}$, which becomes $\tilde{\wp}_\jmath^* = \tilde{\wp}_\jmath \ \forall \ \jmath = 1, 2, 3, 4$
Without loss of generality, we assume that $\tilde{\wp}_2^* \geq \tilde{\wp}_1^* \geq \tilde{\wp}_4^* \geq \tilde{\wp}_3^*$
i.e., $\tilde{\wp}_{\sigma(1)}^* \geq \tilde{\wp}_{\sigma(2)}^* \geq \tilde{\wp}_{\sigma(3)}^* \geq \tilde{\wp}_{\sigma(4)}^*$.

Therefore, $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) = \bigotimes_{\jmath=1}^{4} (\tilde{\wp}_{\sigma(\jmath)}^*)^{\ell_\jmath} = (\tilde{\wp}_{\sigma(1)}^*)^{\ell_1} \otimes (\tilde{\wp}_{\sigma(2)}^*)^{\ell_2} \otimes (\tilde{\wp}_{\sigma(3)}^*)^{\ell_3} \otimes (\tilde{\wp}_{\sigma(4)}^*)^{\ell_4}$

$$= \bigotimes_{\jmath=1}^{4} (\tilde{\wp}_{\sigma(\jmath)})^{\ell_\jmath} = PyFOWG_\ell(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4).$$

Theorem 2.5 If $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\hbar)^T = (\frac{1}{\hbar}, \frac{1}{\hbar}, ..., \frac{1}{\hbar})^T$ then $PyFHG_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$

$= PyFOWG_{\ell}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_{\hbar})$.

Proof: The proof is similar as Theorem 5.12.

It is clear from the Theorem 2.4 and Theorem 2.5 that PyFWG and PyFOWG operators are the particular cases of the PyFHG operator, or it can be said that the PyFHG operator is the generalization of PyFWG and PyFOWG operators.

3 Hamacher t-norm and Hamacher t-conorm

Hamacher introduced the Hamacher t-norm and Hamacher t-conorm operators. Liu and Peide [34] proposed the de

nitions of the same, Lu et al. [64] as follows:

Definition 3.1 (Hamacher t-norm operator and Hamacher t-conorm operator)
Hamacher t-norm is a function $\otimes^H : [0,1]^2 \rightarrow [0,1]$ which is defined below.
$a \otimes^H b = \frac{ab}{\kappa+(1-\kappa)(a+b-ab)}$ for all $(a,b) \in [0,1]^2$ and scalar parameter $\kappa > 0$.

Hamacher t-conorm operator is a function $\oplus^H : [0,1]^2 \rightarrow [0,1]$ which is defined below.
$a \oplus^H b = \frac{a+b-ab-(1-\kappa)ab}{1-(1-\kappa)ab}$ $\forall\,(a,b) \in [0,1]^2$ and scalar parameter $\kappa > 0$.

Case-1: If $\kappa = 1$ then $a \otimes^H b = ab$, which is basic algebraic t-norm operator and $a \oplus^H b = a + b - ab$, which is basic algebraic t-conorm operator.

Case-2: If $\kappa = 2$ then $a \otimes^H b = \frac{ab}{1+(1-a)(1-b)}$, which is called Einstein t-norm operator and $a \oplus^H b = \frac{a+b}{1+ab}$, which is called Einstein t-conorm operator.

Example 1 Let $a = 0.7, b = 0.4$ and $\kappa = 2$ then

$$a \otimes^H b = \frac{0.7 \times 0.4}{2 + (1-2)(0.7 + 0.4 - 0.28)} = 0.237 \in [0,1]$$

$$a \oplus^H b = \frac{0.7 + 0.4 - 0.4 \times 0.7 - (1-2) \times 0.7 \times 0.4}{1 - (1-2) \times 0.7 \times 0.4} = 0.781 \in [0,1]$$

We have applied Hamacher t-norm and Hamacher t-conorm on real numbers on [0, 1], but now we are going to apply those operators on PyFNs and with the help of Hamacher t-norm and t-conorm operations [34, 36] and basic Pythagorean fuzzy ORs [49, 52] we get the following ORs which are defined as follows:

3.1 ORs on Pythagorean Fuzzy Numbers Based on Hamacher t-norm and Hamacher t-conorm Operators

Let $\tilde{\wp} = \langle \mathfrak{T}_{\tilde{\wp}}, \aleph_{\tilde{\wp}} \rangle$, $\tilde{\wp}_1 = \langle \mathfrak{T}_{\tilde{\wp}_1}, \aleph_{\tilde{\wp}_1} \rangle$ and $\tilde{\wp}_2 = \langle \mathfrak{T}_{\tilde{\wp}_2}, \aleph_{\tilde{\wp}_2} \rangle$ be three PyFNs in PyFN($\mathfrak{U}$) and $\tau > 0$ be an any scalar. The basic ORs [55] are as follows:

(i) $\tilde{\wp}_1 \oplus^H \tilde{\wp}_2 = \left\langle \sqrt{\frac{\mathfrak{T}^2_{\tilde{\wp}_1} + \mathfrak{T}^2_{\tilde{\wp}_2} - \mathfrak{T}^2_{\tilde{\wp}_1}\mathfrak{T}^2_{\tilde{\wp}_2} - (1-\kappa)\mathfrak{T}^2_{\tilde{\wp}_1}\mathfrak{T}^2_{\tilde{\wp}_2}}{1-(1-\kappa)\mathfrak{T}^2_{\tilde{\wp}_1}\mathfrak{T}^2_{\tilde{\wp}_2}}}, \frac{\aleph_{\tilde{\wp}_1}\aleph_{\tilde{\wp}_2}}{\sqrt{\kappa+(1-\kappa)(\aleph^2_{\tilde{\wp}_1} + \aleph^2_{\tilde{\wp}_2} - \aleph^2_{\tilde{\wp}_1}\aleph^2_{\tilde{\wp}_2})}} \right\rangle$.

(ii) $\tilde{\wp}_1 \otimes^H \tilde{\wp}_2 = \left\langle \frac{\mathfrak{T}_{\tilde{\wp}_1}\mathfrak{T}_{\tilde{\wp}_2}}{\sqrt{\kappa+(1-\kappa)(\mathfrak{T}^2_{\tilde{\wp}_1} + \mathfrak{T}^2_{\tilde{\wp}_2} - \mathfrak{T}^2_{\tilde{\wp}_1}\mathfrak{T}^2_{\tilde{\wp}_2})}}, \sqrt{\frac{\aleph^2_{\tilde{\wp}_1} + \aleph^2_{\tilde{\wp}_2} - \aleph^2_{\tilde{\wp}_1}\aleph^2_{\tilde{\wp}_2} - (1-\kappa)\aleph^2_{\tilde{\wp}_1}\aleph^2_{\tilde{\wp}_2}}{1-(1-\kappa)\aleph^2_{\tilde{\wp}_1}\aleph^2_{\tilde{\wp}_2}}} \right\rangle$.

(iii) $\tau\tilde{\wp} = \left\langle \sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}}\}^{\tau} - (1-\mathfrak{T}^2_{\tilde{\wp}})^{\tau}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}}\}^{\tau} + (\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}})^{\tau}}}, \frac{\sqrt{\kappa}(\aleph^{\tau}_{\tilde{\wp}})}{\sqrt{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}})\}^{\tau} + (\kappa-1)\aleph^{2\tau}_{\tilde{\wp}}}} \right\rangle$.

(iv) $\tilde{\wp}^{\tau} = \left\langle \frac{\sqrt{\kappa}(\mathfrak{T}^{\tau}_{\tilde{\wp}})}{\sqrt{\{1+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}})\}^{\tau} + (\kappa-1)\mathfrak{T}^{2\tau}_{\tilde{\wp}}}}, \sqrt{\frac{\{1+(\kappa-1)\aleph^2_{\tilde{\wp}}\}^{\tau} - (1-\aleph^2_{\tilde{\wp}})^{\tau}}{\{1+(\kappa-1)\aleph^2_{\tilde{\wp}}\}^{\tau} + (\kappa-1)(1-\aleph^2_{\tilde{\wp}})^{\tau}}} \right\rangle$.

4 Pythagorean Fuzzy Hamacher Averaging Operators

In this section, three types of Pythagorean fuzzy Hamacher averaging operators have been discussed, which are the Pythagorean fuzzy Hamacher weighted averaging (PyFHWA) operator, Pythagorean fuzzy Hamacher ordered weighted averaging (PyFHOWA) operator and Pythagorean fuzzy Hamacher hybrid averaging (PyFHHA) operator. Now $\bigoplus^H$ and $\bigotimes^H$ are denoted as $\bigoplus$ and $\bigotimes$, respectively, and the definitions of those operators are as follows:

4.1 Pythagorean Fuzzy Hamacher Weighted Averaging Operator

Definition 4.1 (Pythagorean fuzzy Hamacher weighted averaging (PyFHWA) operator)

Let $\mathfrak{P} = \{\tilde{\wp}_{\jmath} = \langle \mathfrak{T}_{\tilde{\wp}_{\jmath}}, \aleph_{\tilde{\wp}_{\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHWA operator is a mapping $PyFHWA_{\ell} : \mathfrak{P}^{\hbar} \rightarrow \mathfrak{P}$, defined below.

$PyFHWA_{\ell}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_{\hbar}) = \bigoplus_{\jmath=1}^{\hbar} (\ell_{\jmath}\tilde{\wp}_{\jmath})$, where, $\ell=(\ell_1,\ell_2,...,\ell_\hbar)^T$ is a weight vector such that $\ell_\jmath\in[0,1], \jmath=1,2,...,\hbar$ and $\sum_{\jmath=1}^{\hbar}\ell_\jmath=1$.

Theorem 4.1 Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle: \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). Then prove that $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2\oplus...\oplus\ell_\hbar\tilde{\wp}_\hbar$

$$=\left\langle\sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}},\right.$$
$$\left.\frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}}\right\rangle$$

Proof: For $\hbar=2$, we have $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2$.
Now, $\ell_1\tilde{\wp}_1=$
$$\left\langle\sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_1}\}^{\ell_1}-(1-\mathfrak{T}^2_{\tilde{\wp}_1})^{\ell_1}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_1}\}^{\ell_1}+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}_1})^{\ell_1}}},\frac{\sqrt{\kappa}\aleph^{\ell_1}_{\tilde{\wp}_1}}{\sqrt{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_1})\}^{\ell_1}+(\kappa-1)\aleph^{2\ell_1}_{\tilde{\wp}_1}}}\right\rangle$$
and $\ell_2\tilde{\wp}_2$=
$$\left\langle\sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_2}\}^{\ell_2}-(1-\mathfrak{T}^2_{\tilde{\wp}_2})^{\ell_2}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_2}\}^{\ell_2}+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}_2})^{\ell_2}}},\frac{\sqrt{\kappa}\aleph^{\ell_2}_{\tilde{\wp}_2}}{\sqrt{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_2})\}^{\ell_2}+(\kappa-1)\aleph^{2\ell_2}_{\tilde{\wp}_2}}}\right\rangle$$
$$\therefore \ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2=\left\langle\sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_1}\}^{\ell_1}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_2}\}^{\ell_2}-(1-\mathfrak{T}^2_{\tilde{\wp}_1})^{\ell_1}(1-\mathfrak{T}^2_{\tilde{\wp}_2})^{\ell_2}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_1}\}^{\ell_1}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_2})\}^{\ell_2}+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}_1})^{\ell_1}(1-\mathfrak{T}^2_{\tilde{\wp}_2})^{\ell_2}}},\right.$$
$$\left.\frac{\sqrt{\kappa}\aleph^{\ell_1}_{\tilde{\wp}_1}\aleph^{\ell_2}_{\tilde{\wp}_2}}{\sqrt{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_1})\}^{\ell_1}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_2})\}^{\ell_2}+(\kappa-1)\aleph^{2\ell_1}_{\tilde{\wp}_1}\aleph^{2\ell_2}_{\tilde{\wp}_2}}}\right\rangle.$$
$i.e., PyFHWA(\tilde{\wp}_1,\tilde{\wp}_2)=\ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2=$
$$\left\langle\sqrt{\frac{\prod_{\jmath=1}^{2}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}-\prod_{\jmath=1}^{2}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{2}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{2}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}},\right.$$
$$\left.\frac{\sqrt{\kappa}\prod_{\jmath=1}^{2}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{2}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{2}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}}\right\rangle.$$
That is, the theorem is valid for $\hbar=2$.
We assume that the theorem is true for $\hbar=\varsigma\in\mathbf{N}$ $i.e.,$
$PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\varsigma)=\ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2\oplus...\oplus\ell_\varsigma\tilde{\wp}_\varsigma$=
$$\left\langle\sqrt{\frac{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\varsigma}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\varsigma}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}},\right.$$
$$\left.\frac{\sqrt{\kappa}\prod_{\jmath=1}^{\varsigma}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\varsigma}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}}\right\rangle$$

$\therefore PyFHWA_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\varsigma}, \tilde{\wp}_{\varsigma+1}) = \ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus ... \oplus \ell_\varsigma\tilde{\wp_\varsigma} \oplus \ell_{\varsigma+1}\tilde{\wp}_{\varsigma+1}$

$$= PyFHWA(\ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus ... \oplus \ell_\varsigma\tilde{\wp_\varsigma}) \oplus \ell_{\varsigma+1}\tilde{\wp}_{\varsigma+1}$$

$$=\left\langle \sqrt{\frac{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} - \prod_{\jmath=1}^{\varsigma}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\varsigma}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}}, \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\varsigma}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\varsigma}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\varsigma}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}} \right\rangle \oplus \left\langle \sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\varsigma+1}}\}^{\ell_{\varsigma+1}} - (1-\mathfrak{T}^2_{\tilde{\wp}_{\varsigma+1}})^{\ell_{\varsigma+1}}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\varsigma+1}}\}^{\ell_{\varsigma+1}} + (\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}_{\varsigma+1}})^{\ell_{\varsigma+1}}}}, \frac{\sqrt{\kappa}\cdot\aleph^{\ell_{\varsigma+1}}_{\tilde{\wp}_{\varsigma+1}}}{\sqrt{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_{\varsigma+1}})\}^{\ell_{\varsigma+1}} + (\kappa-1)\aleph^{2\ell_{\varsigma+1}}_{\tilde{\wp}_{\varsigma+1}}}} \right\rangle =$$

$$\left\langle \sqrt{\frac{\prod_{\jmath=1}^{\varsigma+1}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} - \prod_{\jmath=1}^{\varsigma+1}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\varsigma+1}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\varsigma+1}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}}, \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\varsigma+1}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\varsigma+1}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\varsigma+1}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}} \right\rangle.$$

Hence the theorem is true for $\hbar = \varsigma + 1$ when it is assumed to be true for $\hbar = \varsigma$. It is also proved that the theorem is true for $\hbar = 2$. Then by Mathematical induction, we can say that the theorem is true for all $\hbar \in \mathbf{N}$.
Therefore, $PyFHWA_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus ... \oplus \ell_\hbar\tilde{\wp_\hbar}$

$$=\left\langle \sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} - \prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}}, \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}} \right\rangle$$

for all $\hbar \in \mathbf{N}$. It can also be proved that the resultant number is also a PyFN.
The value of this operator concerning some PyFNs is shown in Example 2.

Example 2 Let $\tilde{\wp_1} = \langle 0.7, 0.6\rangle$, $\tilde{\wp_2} = \langle 0.5, 0.4\rangle$, $\tilde{\wp_3} = \langle 0.7, 0.3\rangle$, $\tilde{\wp_4} = \langle 0.3, 0.4\rangle$, $\tilde{\wp_4} = \langle 0.3, 0.4\rangle$ be the four PyFNs with the weight vector $\ell = (0.2, 0.1, 0.3, 0.4)^T$ and for $\kappa = 3$,
$PyFHWA_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, \tilde{\wp_3}, \tilde{\wp_4}) = \ell_1\tilde{\wp_1} \oplus \ell_2\tilde{\wp_2} \oplus \ell_3\tilde{\wp_3} \oplus \ell_4\tilde{\wp_4}$

$$=\left\langle \sqrt{\frac{\prod_{\jmath=1}^{4}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} - \prod_{\jmath=1}^{4}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{4}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{4}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}}, \right.$$

$$\frac{\sqrt{\kappa}\prod_{\jmath=1}^{4}\aleph_{\tilde{\wp}_\jmath}^{\ell_\jmath}}{\sqrt{\prod_{\jmath=1}^{4}\{1+(\kappa-1)(1-\aleph_{\tilde{\wp}_\jmath}^2)\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{4}\aleph_{\tilde{\wp}_\jmath}^{2\ell_\jmath}}}\Bigg\rangle=\langle 0.497, 0.422\rangle.$$

Theorem 4.2 (Idempotency Property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). If $\tilde{\wp}_\jmath=\tilde{\wp}$ for all $\jmath=1,2,...,\hbar$ then $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\tilde{\wp}$.

Proof: We have $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\ell_1\tilde{\wp}_1\oplus\ell_2\tilde{\wp}_2\oplus...\oplus\ell_\hbar\tilde{\wp}_\hbar$

$$=\Bigg\langle\sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}_\jmath}^2\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}_{\tilde{\wp}_\jmath}^2)^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}_\jmath}^2\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}_{\tilde{\wp}_\jmath}^2)^{\ell_\jmath}}},$$

$$\frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\aleph_{\tilde{\wp}_\jmath}^{\ell_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph_{\tilde{\wp}_\jmath}^2)\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\aleph_{\tilde{\wp}_\jmath}^{2\ell_\jmath}}}\Bigg\rangle$$

As $\tilde{\wp}_\jmath=\tilde{\wp}$ for all $\jmath$=1,2,...,$\hbar$ $i.e$ $\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle=\langle\mathfrak{T}_{\tilde{\wp}},\aleph_{\tilde{\wp}}\rangle$ for all $\jmath=1,2,...,\hbar$ then

$PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)$=

$$\Bigg\langle\sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}}^2\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}-(1-\mathfrak{T}_{\tilde{\wp}}^2)^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}}^2\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}+(\kappa-1)(1-\mathfrak{T}_{\tilde{\wp}}^2)^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}},\frac{\sqrt{\kappa}\cdot(\aleph_{\tilde{\wp}})^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}{\sqrt{\{1+(\kappa-1)(1-\aleph_{\tilde{\wp}}^2)\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}+(\kappa-1)(\aleph_{\tilde{\wp}})^{2\sum_{\jmath=1}^{\hbar}\ell_\jmath}}}\Bigg\rangle$$

As we have $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)$=

$$\Bigg\langle\sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}}^2\}-(1-\mathfrak{T}_{\tilde{\wp}}^2)}{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}}^2\}+(\kappa-1)(1-\mathfrak{T}_{\tilde{\wp}}^2}}),\frac{\sqrt{\kappa}\cdot(\aleph_{\tilde{\wp}})}{\sqrt{\{1+(\kappa-1)(1-\aleph_{\tilde{\wp}}^2)\}+(\kappa-1)(\aleph_{\tilde{\wp}})^2}}\Bigg\rangle=\langle\mathfrak{T}_{\tilde{\wp}},\aleph_{\tilde{\wp}}\rangle=\tilde{\wp}.$$

Theorem 4.3 (Boundness Property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$) with the total order relation $\leq$ defined on it. If $\tilde{\wp}^+=\langle\mathfrak{T}_{\tilde{\wp}^+},\aleph_{\tilde{\wp}^+}\rangle=\max_\jmath\{\tilde{\wp}_\jmath\}=\max_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ and $\tilde{\wp}^-=\langle\mathfrak{T}_{\tilde{\wp}^-},\aleph_{\tilde{\wp}^-}\rangle=\min_\jmath\{\tilde{\wp}_\jmath\}=\min_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ then $PyFHWA_\ell\colon\mathfrak{P}^\hbar\to\mathfrak{P}$ is bounded as below.
$\tilde{\wp}^-\leq PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)\leq\tilde{\wp}^+$.

Proof: As $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)$ is also a PyFN then we assume that it is $\langle\mathfrak{T}_{\tilde{\wp}},\aleph_{\tilde{\wp}}\rangle$. It is given that $\tilde{\wp}^+=\langle\mathfrak{T}_{\tilde{\wp}^+},\aleph_{\tilde{\wp}^+}\rangle=\max_\jmath\{\tilde{\wp}_\jmath\}=\max_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ and $\tilde{\wp}^-=\langle\mathfrak{T}_{\tilde{\wp}^-},\aleph_{\tilde{\wp}^-}\rangle=\min_\jmath\{\tilde{\wp}_\jmath\}=\min_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ then $\mathfrak{T}_{\tilde{\wp}_\jmath}\leq\mathfrak{T}_{\tilde{\wp}^+}$ and $\aleph_{\tilde{\wp}_\jmath}\geq\aleph_{\tilde{\wp}^+}$ for all $\jmath=1,2,...,\hbar$.
And $\mathfrak{T}_{\tilde{\wp}_\jmath}\geq\mathfrak{T}_{\tilde{\wp}^-}$ and $\aleph_{\tilde{\wp}_\jmath}\leq\aleph_{\tilde{\wp}^-}$ for all $\jmath=1,2,...,\hbar$.
Let $f(\varrho)=\frac{1+(\kappa-1)\varrho^2}{1-\varrho^2}$
$\therefore f'(\varrho)=\frac{2x\kappa}{(1-\varrho^2)^2}>0$, for all $\varrho\in(0,1]$ and $\kappa>0$, which shows that $f(\varrho)$ is increasing function.
Therefore, $\mathfrak{T}_{\tilde{\wp}_\jmath}\leq\mathfrak{T}_{\tilde{\wp}^+}$ we can write $f(\mathfrak{T}_{\tilde{\wp}_\jmath})\leq f(\mathfrak{T}_{\tilde{\wp}^+})$ $i.e.,$ $\frac{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}_\jmath}^2}{1-\mathfrak{T}_{\tilde{\wp}_\jmath}^2}\leq\frac{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}^+}^2}{1-\mathfrak{T}_{\tilde{\wp}^+}^2}$ or, $\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}_\jmath}^2\}^{\ell_\jmath}}{(1-\mathfrak{T}_{\tilde{\wp}_\jmath}^2)^{\ell_\jmath}}\leq\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}^+}^2\}^{\ell_\jmath}}{(1-\mathfrak{T}_{\tilde{\wp}^+}^2)^{\ell_\jmath}}$ or, $\prod_{\jmath=1}^{\hbar}\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}_\jmath}^2\}^{\ell_\jmath}}{(1-\mathfrak{T}_{\tilde{\wp}_\jmath}^2)^{\ell_\jmath}}\leq\prod_{\jmath=1}^{\hbar}\frac{\{1+(\kappa-1)\mathfrak{T}_{\tilde{\wp}^+}^2\}^{\ell_\jmath}}{(1-\mathfrak{T}_{\tilde{\wp}^+}^2)^{\ell_\jmath}}$ then

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}} \leq \frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\ell_\jmath}} \text{ therefore,}$$

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}+\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}} \leq \frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\ell_\jmath}+\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\ell_\jmath}} \text{ or,}$$

$$\sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_\jmath}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})^{\ell_\jmath}}}$$

$$\leq \sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\ell_\jmath}}}$$

$$\therefore \mathfrak{T}_{\tilde{\wp}} \leq \sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}-(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}+})^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}} \; i.e., \mathfrak{T}_{\tilde{\wp}} \leq \sqrt{\frac{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}-(1-\mathfrak{T}^2_{\tilde{\wp}+})}{\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}+}\}+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}+})}}$$

$\therefore \mathfrak{T}_{\tilde{\wp}} \leq \mathfrak{T}_{\tilde{\wp}^+}$, Similarly, we can prove that $\mathfrak{T}_{\tilde{\wp}} \geq \mathfrak{T}_{\tilde{\wp}^-}$.
Let $g(\varrho) = \frac{1+(\kappa-1)(1-\varrho^2)}{\varrho^2}$
therefore $g'(\varrho) = -\frac{2\kappa}{\varrho^3} < 0$ for all $\varrho \in (0,1]$ and $\kappa > 0$, which shows that $g(\varrho)$ is decreasing function. Hence for $\aleph_{\tilde{\wp}_\jmath} \geq \aleph_{\tilde{\wp}^+}$ we have $g(\aleph_{\tilde{\wp}_\jmath}) \leq g(\aleph_{\tilde{\wp}^+})$.
$i.e., \frac{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})}{\aleph^2_{\tilde{\wp}_\jmath}} \leq \frac{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}+})}{\aleph^2_{\tilde{\wp}+}}$ then

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}} \leq \frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}+})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}+}} \text{ or,}$$

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}+(\kappa-1) \leq \frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}+})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}+}}+(\kappa-1) \text{ or,}$$

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}+(\kappa-1) \leq \frac{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}+})\}^{\sum_{\jmath=1}^{\hbar}\ell_\jmath}}{\aleph_{\tilde{\wp}+}^{2\sum_{\jmath=1}^{\hbar}\ell_\jmath}}+(\kappa-1)$$

or,

$$\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}+(\kappa-1)\leq\frac{\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}^+})\}}{\aleph^2_{\tilde{\wp}^+}}+(\kappa-1) \text{ or,}$$

$$\sqrt{\frac{\kappa\cdot\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath})}}\geq\aleph_{\tilde{\wp}^+} \ i.e.,$$

$$\frac{\sqrt{\kappa}\cdot\prod_{\jmath=1}^{\hbar}\aleph^{\ell_\jmath}_{\tilde{\wp}_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_\jmath})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(\aleph^{2\ell_\jmath}_{\tilde{\wp}_\jmath})}}\geq\aleph_{\tilde{\wp}^+}$$ then $\aleph_{\tilde{\wp}}\geq\aleph_{\tilde{\wp}^+}$ and similarly we can show that $\aleph_{\tilde{\wp}}\leq\aleph_{\tilde{\wp}^-}$.

Thus we have proved that $\mathfrak{T}_{\tilde{\wp}^-}\leq\mathfrak{T}_{\tilde{\wp}}\leq\mathfrak{T}_{\tilde{\wp}^+}$ and $\aleph_{\tilde{\wp}^-}\geq\aleph_{\tilde{\wp}}\geq\aleph_{\tilde{\wp}^+}$.
Now $Sc(\tilde{\wp})=\mathfrak{T}^2_{\tilde{\wp}}-\aleph^2_{\tilde{\wp}}\leq\mathfrak{T}^2_{\tilde{\wp}^+}-\aleph^2_{\tilde{\wp}^+}=Sc(\tilde{\wp}^+)$ and if $Sc(\tilde{\wp})<Sc(\tilde{\wp}^+)$ then $\tilde{\wp}<\tilde{\wp}^+$ and if equality occurs then with the help of Lemma 2.1, we can write that $Ac(\tilde{\wp})=Ac(\tilde{\wp}^+)$ and then $\tilde{\wp}=\tilde{\wp}^+$. Hence after calculating score values and accuracy values, we can write that $\tilde{\wp}\leq\tilde{\wp}^+$.
Again $Sc(\tilde{\wp})=\mathfrak{T}^2_{\tilde{\wp}}-\aleph^2_{\tilde{\wp}}\geq\mathfrak{T}^2_{\tilde{\wp}^-}-\aleph^2_{\tilde{\wp}^-}=Sc(\tilde{\wp}^-)$ and if $Sc(\tilde{\wp})>Sc(\tilde{\wp}^-)$ then $\tilde{\wp}>\tilde{\wp}^-$. If equality occurs then with the help of Lemma 2.1, we can write that $Ac(\tilde{\wp})=Ac(\tilde{\wp}^-)$ and then $\tilde{\wp}=\tilde{\wp}^-$. Hence calculating score and accuracy values we can write that $\tilde{\wp}^-\leq\tilde{\wp}$.
Then we can write that $\tilde{\wp}^-\leq\tilde{\wp}\leq\tilde{\wp}^+$.
Hence, $\tilde{\wp}^-\leq PyFHWA(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)\leq\tilde{\wp}^+$.

Theorem 4.4 (Monotonicity Property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath:\jmath=1,2,...,\hbar\}$ and $\mathfrak{P}'=\{\tilde{\wp}_\jmath{}':\jmath=1,2,...,\hbar\}$ be the two sets of PyFNs. If $\tilde{\wp}_\jmath\leq\tilde{\wp}_\jmath{}'$ then $PyFHWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)\leq PyFHWA_\ell(\tilde{\wp}_1{}',\tilde{\wp}_2{}',...,\tilde{\wp}_\hbar{}')$.
The Monotonicity property of the PyFHWA operator is verified by Example 3.

Example 3

Let $\tilde{\wp}_1=\langle0.2,0.7\rangle$,$\tilde{\wp}_2=\langle0.3,0.9\rangle$,$\tilde{\wp}_3=\langle0.5,0.8\rangle$ and $\tilde{\wp}_1{}'=\langle0.3,0.6\rangle$,$\tilde{\wp}_2{}'=\langle0.4,0.3\rangle$,$\tilde{\wp}_3{}'=\langle0.6,0.7\rangle$. Let the weight vector be $\ell=(0.2,0.3,0.5)^T$ and parameter value $\kappa=3$ then

in this case $\tilde{\wp}_1\leq_L\tilde{\wp}_1{}'$, $\tilde{\wp}_2\leq_L\tilde{\wp}_2{}'$, $\tilde{\wp}_3\leq_L\tilde{\wp}_3{}'$. Hence $\tilde{\wp}_1\leq\tilde{\wp}_1{}'$, $\tilde{\wp}_2\leq\tilde{\wp}_2{}'$, $\tilde{\wp}_3\leq\tilde{\wp}_3{}'$. It is given that $\kappa=3$ and $\ell=(0.2,0.3,0.5)^T$. Now $PyFHWA(\tilde{\wp}_1,\tilde{\wp}_2,\tilde{\wp}_3)=\langle0.3614,0.8221\rangle$ and $PyFHWA(\tilde{\wp}_1{}',\tilde{\wp}_2{}',\tilde{\wp}_3{}')=\langle0.4684,0.7172\rangle$
Now $Sc(PyFHWA(\tilde{\wp}_1,\tilde{\wp}_2,\tilde{\wp}_3))=-0.5451$ and $Sc(PyFHWA(\tilde{\wp}_1{}',\tilde{\wp}_2{}',\tilde{\wp}_3{}'))=-0.2950$.
$\therefore PyFHWA(\tilde{\wp}_1,\tilde{\wp}_2,\tilde{\wp}_3)\leq PyFHWA(\tilde{\wp}_1{}',\tilde{\wp}_2{}',\tilde{\wp}_3{}')$.

4.2 Pythagorean Fuzzy Hamacher Ordered Weighted Averaging Operator

Definition 4.2 (Pythagorean fuzzy Hamacher ordered weighted averaging (PyFHOWA) operator)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle:\jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHOWA operator is a mapping $PyFHOWA_\ell:\mathfrak{P}^\hbar\to\mathfrak{P}$ which is defined below.

$$PyFHOWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\bigoplus_{\jmath=1}^{\hbar}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}),$$

where $\ell=(\ell_1,\ell_2,...,\ell_\hbar)^T$ is a weight vector such that $\ell_\jmath\in[0,1]$, $\jmath=1,2,...,\hbar$ and $\sum_{\jmath=1}^{\hbar}\ell_\jmath=1$ and $(\sigma(1),\sigma(2),...,\sigma(\hbar))$ is a permutation of (1,2,...,$\hbar$) such that $\tilde{\wp}_{\sigma(\jmath-1)}\geq\tilde{\wp}_{\sigma(\jmath)}\ \forall\ \jmath=2,3,...,\hbar$.

Theorem 4.5 Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle:\jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). Then

$$PyFHOWA_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\bigoplus_{\jmath=1}^{\hbar}(\ell_\jmath\tilde{\wp_{\sigma(\jmath)}})=\ell_1\tilde{\wp_{\sigma(1)}}\oplus\ell_2\tilde{\wp_{\sigma(2)}}\oplus...\oplus\ell_\hbar\tilde{\wp_{\sigma(\hbar)}}=$$

$$\left\langle \sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma(\jmath)}}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma(\jmath)}})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma(\jmath)}}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma(\jmath)}})^{\ell_\jmath}}}, \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\aleph^{\ell_\jmath}_{\tilde{\wp}_{\sigma(\jmath)}}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_{\sigma(\jmath)}})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\aleph^{2\ell_\jmath}_{\tilde{\wp}_{\sigma(\jmath)}}}} \right\rangle.$$

Theorem 4.6 (Idempotency property)
Let $\tilde{\mathfrak{P}} = \{\tilde{\wp}_\jmath : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs on PyFN($\mathfrak{U}$). If $\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle = \tilde{\wp} = \langle \mathfrak{T}_{\tilde{\wp}}, \aleph_{\tilde{\wp}}\rangle$ for all $\jmath = 1, 2, ..., \hbar$ then $PyFHOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \tilde{\wp}$.

Theorem 4.7 (Boundness property)

Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$) with the total order relation $\leq$ defined on it. If $\tilde{\wp}^+ = \langle \mathfrak{T}_{\tilde{\wp}^+}, \aleph_{\tilde{\wp}^+}\rangle = \max_\jmath\{\tilde{\wp}_\jmath\} = \max_\jmath\{\langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle\}$ and $\tilde{\wp}^- = \langle \mathfrak{T}_{\tilde{\wp}^-}, \aleph_{\tilde{\wp}^-}\rangle = \min_\jmath\{\tilde{\wp}_\jmath\} = \min_\jmath\{\langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle\}$ then $PyFHOWA_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ is bounded as below.
$\tilde{\wp}^- \leq PyFHOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) \leq \tilde{\wp}^+$.

Theorem 4.8 (Monotonicity property)

Let $\mathfrak{P} = \{\tilde{\wp}_\jmath : \jmath = 1, 2, ..., \hbar\}$ and $\mathfrak{P}' = \{\tilde{\wp}_\jmath{}' : \jmath = 1, 2, ..., \hbar\}$ be the two sets PyFNs. If $\tilde{\wp}_\jmath \leq \tilde{\wp}_\jmath{}'$ then $PyFHOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) \leq PyFHOWA_\ell(\tilde{\wp}_1{}', \tilde{\wp}_2{}', ..., \tilde{\wp}_\hbar{}')$.

Theorem 4.9 (Commutativity property)

Let $\mathfrak{P} = \{\tilde{\wp}_\jmath : \jmath = 1, 2, ..., \hbar\}$ and $\mathfrak{P}' = \{\tilde{\wp}_\jmath{}' : \jmath = 1, 2, ..., \hbar\}$ be the two sets of PyFNs. If $(\tilde{\wp}_1{}', \tilde{\wp}_2{}', ..., \tilde{\wp}_\hbar{}')$ be any permutation of $(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar)$ then $PyFHOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = PyFHOWA_\ell(\tilde{\wp}_1{}', \tilde{\wp}_2{}', ..., \tilde{\wp}_\hbar{}')$.
Pythagorean fuzzy Hamacher hybrid averaging (PyFHHA) operator is a combination of PyFHWA and PyFHOWA operators, and it is defined below.

4.3 Pythagorean Fuzzy Hamacher Hybrid Averaging Operator

Definition 4.3 (Pythagorean fuzzy Hamacher hybrid averaging (PyFHHA) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHHA operator is a mapping $PyFHHA_{\ell,\Omega} : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFHHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \bigoplus_{\jmath=1}^{\hbar}(\ell_\jmath \tilde{\wp}^*_{\sigma(\jmath)}),$$

where $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar}\ell_\jmath = 1$ and $\tilde{\wp}^*_\jmath = \hbar\Omega_\jmath\tilde{\wp}_\jmath$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of $(1, 2, ..., \hbar)$ such that $\tilde{\wp}^*_{\sigma(\jmath-1)} \geq \tilde{\wp}^*_{\sigma(\jmath)}\ \forall\ \jmath = 2, 3, ..., \hbar$ and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\hbar)^T$ is a associated weight vector such that $\Omega_\jmath \in [0, 1]\ \forall\ \jmath = 1, 2, 3, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar}\Omega_\jmath = 1$.
i.e., $PyFHHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \ell_1\tilde{\wp}^*_{\sigma(1)} \oplus \ell_2\tilde{\wp}^*_{\sigma(2)} \oplus ... \oplus \ell_\hbar\tilde{\wp}^*_{\sigma(\hbar)}$.

Theorem 4.10 Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$) then
$PyFHHA_{\ell,\Omega}(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\hbar) = \ell_1\tilde{\wp}^*_{\sigma(1)} \oplus \ell_2\tilde{\wp}^*_{\sigma(2)} \oplus ... \oplus \ell_\hbar\tilde{\wp}^*_{\sigma(\hbar)} =$

$$\left\langle \sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}})^{\ell_\jmath}}},\right.$$

$$\frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\aleph_{\tilde{\wp}_{\sigma^*(\jmath)}}^{\ell_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\aleph_{\tilde{\wp}_{\sigma^*(\jmath)}}^{2})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\aleph_{\tilde{\wp}_{\sigma^*(\jmath)}}^{2\ell_\jmath}}}\Bigg\rangle.$$

Theorem 4.11 If $\ell=(\frac{1}{\hbar},\frac{1}{\hbar},...,\frac{1}{\hbar})^T$ then $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar})=PyFHWA_{\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar})$.

Proof. We prove this theorem by Mathematical induction. Suppose $\ell=(\ell_1,\ell_2)=(\frac{1}{2},\frac{1}{2})^T$ and $\Omega=(\Omega_1,\Omega_2)^T$ be an associated weight vector.
In this case we know that $\tilde{\wp}_\jmath^*=2\Omega_\jmath\tilde{\wp}_\jmath$ for $\jmath=1,2$.
Without loss of generality, we assume that $\tilde{\wp}_2^*\geq\tilde{\wp}_1^*$ i.e., $\tilde{\wp}_{\sigma(1)}^*\geq\tilde{\wp}_{\sigma(2)}^*$

Now, $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2})=\bigoplus_{\jmath=1}^{2}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)=\ell_1\tilde{\wp}_{\sigma(1)}^*\oplus\ell_2\tilde{\wp}_{\sigma(2)}^*=\dfrac{1}{2}[\tilde{\wp}_{\sigma(1)}^*\oplus\tilde{\wp}_{\sigma(2)}^*]=\dfrac{1}{2}[\tilde{\wp}_2^*\oplus\tilde{\wp}_1^*]=\dfrac{1}{2}[2\Omega_2\tilde{\wp}_2\oplus 2\Omega_1\tilde{\wp}_1]=\bigoplus_{\jmath=1}^{2}(\Omega_\jmath\tilde{\wp}_\jmath)=PyFHWA_\Omega(\tilde{\wp_1},\tilde{\wp_2})$.
Hence, the theorem is true for $\hbar=2$.
Suppose, the theorem is true for first $\varsigma\in\mathbf{N}$ terms.
Then $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})=PyFHWA_{\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})$.
i.e. $\ell_1\tilde{\wp}_{\sigma(1)}^*\oplus\ell_2\tilde{\wp}_{\sigma(2)}^*\oplus,...,\oplus\ell_\varsigma\tilde{\wp}_{\sigma(\varsigma)}^*=\Omega_1\tilde{\wp}_1\oplus\Omega_2\tilde{\wp}_2\oplus,...,\oplus\Omega_\varsigma\tilde{\wp}_\varsigma$ and in this case $\ell=(\ell_1,\ell_2,...,\ell_\varsigma)^T=(\frac{1}{\varsigma},\frac{1}{\varsigma},...,\frac{1}{\varsigma})^T$.

Now, $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})=\bigoplus_{\jmath=1}^{\varsigma+1}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)$

$=\underbrace{\ell_1\tilde{\wp}_{\sigma(1)}^*\oplus\ell_2\tilde{\wp}_{\sigma(2)}^*\oplus,...,\oplus\ell_\varsigma\tilde{\wp}_{\sigma(\varsigma)}^*}_{\varsigma\text{terms}}\oplus\ell_{\varsigma+1}\tilde{\wp}_{\sigma(\varsigma+1)}^*$

$=\Omega_1\tilde{\wp}_1\oplus\Omega_2\tilde{\wp}_2\oplus,...,\oplus\Omega_\varsigma\tilde{\wp}_\varsigma\oplus\ell_{\varsigma+1}\tilde{\wp}_{\sigma(\varsigma+1)}^*$
Obviously, we can say that $\tilde{\wp}_{\sigma(\varsigma+1)}^*=\tilde{\wp}_{\varsigma+1}^*=(\varsigma+1)\Omega_{\varsigma+1}\tilde{\wp}_{\varsigma+1}$ and $\ell_{\varsigma+1}=\frac{1}{\varsigma+1}$.

Hence, we can write that $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})=\bigoplus_{\jmath=1}^{\varsigma+1}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)$

$=\ell_1\tilde{\wp}_{\sigma(1)}^*\oplus\ell_2\tilde{\wp}_{\sigma(2)}^*\oplus,...,\oplus\ell_\varsigma\tilde{\wp}_{\sigma(\varsigma)}^*\oplus\ell_{\varsigma+1}\tilde{\wp}_{\sigma(\varsigma+1)}^*$
$=\Omega_1\tilde{\wp}_1\oplus\Omega_2\tilde{\wp}_2\oplus,...,\oplus\Omega_\varsigma\tilde{\wp}_\varsigma\oplus\Omega_{\varsigma+1}\tilde{\wp}_{\varsigma+1}=PyFHWA_\Omega(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})$.
Thus the theorem is true for $\hbar=\varsigma+1$ when it is supposed to be true for $\hbar=\varsigma$.
Hence the theorem is valid for all $\hbar\in\mathbf{N}$ by Mathematical induction.

Theorem 4.12 If $\Omega=(\frac{1}{\hbar},\frac{1}{\hbar},...,\frac{1}{\hbar})^T$ then $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar})=PyFHOWA_{\ell}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar})$.

Proof. Let $\Omega=(\Omega_1,\Omega_2)^T=(\frac{1}{2},\frac{1}{2})^T$ and $\ell=(\ell_1,\ell_2)^T$ be weight vectors.
Now, $\tilde{\wp}_\jmath^*=4\Omega_\jmath\tilde{\wp}_\jmath$, which becomes $\tilde{\wp}_\jmath^*=\tilde{\wp}_\jmath$ for $\jmath=1,2$.
Without loss of generality we suppose $\tilde{\wp}_2^*\geq\tilde{\wp}_1^*$ i.e. $\tilde{\wp}_{\sigma(1)}^*\geq\tilde{\wp}_{\sigma(2)}^*$.

Now, $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},\tilde{\wp_3},\tilde{\wp_4})=\bigoplus_{\jmath=1}^{2}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)=\ell_1\tilde{\wp}_{\sigma(1)}^*\oplus\ell_2\tilde{\wp}_{\sigma(2)}^*$

$=\ell_1\tilde{\wp}_{\sigma(1)}\oplus\ell_2\tilde{\wp}_{\sigma(2)}=\bigoplus_{\jmath=1}^{2}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)})=PyFHOWA_\ell(\tilde{\wp_1},\tilde{\wp_2})$.
Thus the theorem is true for $\hbar=2$.
We assume that the theorem is true for $\hbar=\varsigma\in\mathbf{N}$. Then we can write that

$PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})=PyFHOWA_\ell(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})$ *i.e.*, $\bigoplus_{\jmath=1}^{\varsigma}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)=\bigoplus_{\jmath=1}^{\varsigma}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)})$ and in this case $\Omega=(\frac{1}{\varsigma},\frac{1}{\varsigma},...,\frac{1}{\varsigma})^T$.

Now, $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})=\bigoplus_{\jmath=1}^{\varsigma}(\ell_\jmath\tilde{\wp}_{\sigma(\jmath)}^*)\oplus\ell_{\varsigma+1}\tilde{\wp}_{\sigma(\varsigma+1)}^*$.It is clear that $\tilde{\wp}_{\sigma(\varsigma+1)}^*=\tilde{\wp}_{\sigma(\varsigma+1)}$ and $\ell_{\varsigma+1}=\frac{1}{\varsigma+1}$.
Hence, we can write that $PyFHHA_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})$

$$= \bigoplus_{\jmath=1}^{\varsigma}(\ell_\jmath \tilde{\wp}_{\sigma(\jmath)}) \oplus \ell_{\varsigma+1}\tilde{\wp}_{\sigma(\varsigma+1)} = \bigoplus_{\jmath=1}^{\varsigma+1}(\ell_\jmath \tilde{\wp}_{\sigma(\jmath)}) = PyFHOWA_\ell(\tilde{\wp}_1, \tilde{\wp}_2, ..., \tilde{\wp}_\varsigma, \tilde{\wp}_{\varsigma+1}).$$

Therefore the theorem is true for $\hbar = \varsigma + 1$ when it is assumed to be true for $\hbar = \varsigma$.
Hence by Mathematical induction, we can say that the Lemma is true for all $\hbar \in \mathbf{N}$.

It is clear from the Theorem 4.11, Theorem 4.12 that PyFHWA and PyFHOWA operators are the particular cases of PyFHHA operator or, in other words, PyFHHA operator is the generalization of PyFHWA and PyFHOWA operators.

The value of the PyFHHA operator concerning some PyFNs is shown in Example 4.

Example 4 Let $\tilde{\wp}_1 = \langle 0.2, 0.9\rangle, \tilde{\wp}_2 = \langle 0.3, 0.7\rangle, \tilde{\wp}_3 = \langle 0.1, 0.9\rangle, \tilde{\wp}_4 = \langle 0.4, 0.6\rangle$.Let $\ell = (0.2, 0.1, 0.3, 0.4)^T$ and $\Omega = (0.1, 0.3, 0.4, 0.2)^T$ be the corresponding weight vectors and $\kappa = 3$.
Now $\tilde{\wp}_1^* = 4\Omega_1\tilde{\wp}_1 = 0.4\tilde{\wp}_1 =$

$$0.4\tilde{\wp}_1 = \left\langle \sqrt{\frac{\{1+(3-1)0.2^2\}^{0.4} - (1-0.2^2)^{0.4}}{\{1+(3-1)0.2^2\}^{0.4} + (3-1)(1-0.2^2)^{0.4}}}, \right.$$

$$\left. \frac{\sqrt{3}\cdot(0.9)^{0.4}}{\sqrt{\{1+(3-1)(1-0.9^2)\}^{0.4} + (3-1)0.9^{(2\times 0.4)}}} \right\rangle$$

$$= \langle 0.126, 0.962\rangle.$$

Similarly, we can find
$\tilde{\wp}_2^* = 4\Omega_2\tilde{\wp}_2 = 1.2\tilde{\wp}_2 = \langle 0.330, 0.593\rangle$.
$\tilde{\wp}_3^* = 4\Omega_3\tilde{\wp}_3 = 1.6\tilde{\wp}_3 = \langle 0.590, 0.878\rangle$.
$\tilde{\wp}_4^* = 4\Omega_4\tilde{\wp}_4 = 0.8\tilde{\wp}_4 = \langle 0.356, 0.686\rangle$.
Now $Sc(\tilde{\wp}_1^*) = -0.910$, $Sc(\tilde{\wp}_2^*) = -0.243$, $Sc(\tilde{\wp}_3^*) = -0.423$, $Sc(\tilde{\wp}_4^*) = -0.344$.
$\therefore \tilde{\wp}_2^* > \tilde{\wp}_4^* > \tilde{\wp}_3^* > \tilde{\wp}_1^*$ then $\tilde{\wp}_{\sigma(1)}^* = \tilde{\wp}_2^*$, $\tilde{\wp}_{\sigma(2)}^* = \tilde{\wp}_4^*$, $\tilde{\wp}_{\sigma(3)}^* = \tilde{\wp}_3^*$ and $\tilde{\wp}_{\sigma(4)}^* = \tilde{\wp}_1^*$
$\therefore PyFHHA(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3, \tilde{\wp}_4) =$

$$\left\langle \sqrt{\frac{\prod_{\jmath=1}^{4}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}}\}^{\ell_\jmath} - \prod_{\jmath=1}^{4}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}})^{\ell_\jmath}}{\prod_{\jmath=1}^{4}\{1+(\kappa-1)\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}}\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{4}(1-\mathfrak{T}^2_{\tilde{\wp}_{\sigma^*(\jmath)}})^{\ell_\jmath}}}, \right.$$

$$\left. \frac{\sqrt{\kappa}\prod_{\jmath=1}^{4}\aleph^{\ell_\jmath}_{\tilde{\wp}_{\sigma^*(\jmath)}}}{\sqrt{\prod_{\jmath=1}^{4}\{1+(\kappa-1)(1-\aleph^2_{\tilde{\wp}_{\sigma^*(\jmath)}})\}^{\ell_\jmath} + (\kappa-1)\prod_{\jmath=1}^{4}\aleph^{2\ell_\jmath}_{\tilde{\wp}_{\sigma^*(\jmath)}}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{0.419}{2.947}}, \frac{\sqrt{3}\times 0.821}{\sqrt{1.503 + 2\times 0.674}} \right\rangle = \langle 0.377, 0.842\rangle.$$

5 Pythagorean Fuzzy Hamacher Geometric Operators

In this section we will discuss about three types of Pythagorean fuzzy Hamacher geometric AOs which are Pythagorean fuzzy Hamacher weighted geometric (PyFHWG) operator, Pythagorean fuzzy Hamacher oedered weighted geometric (PyFHOWG) operator and Pythagorean fuzzy Hamacher hybrid geometric (PyFHHG) operator.

5.1 Pythagorean Fuzzy Hamacher Weighted Geometric Operator

Definition 5.1 (Pythagorean fuzzy Hamacher weighted geometric (PyFHWG) operator)
Let $\mathfrak{P} = \{\tilde{\wp}_\jmath = \langle \mathfrak{T}_{\tilde{\wp}_\jmath}, \aleph_{\tilde{\wp}_\jmath}\rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of Pythagorean fuzzy numbers in PyFN($\mathfrak{U}$). PyFHWG operator is a mapping $PyFHWG_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFHWG_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \bigotimes_{\jmath=1}^{\hbar}(\tilde{\wp_\jmath})^{\ell_\jmath},$$

where $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$.

Theorem 5.1 Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). Then

$$PyFHWG_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \bigotimes_{\jmath=1}^{\hbar}(\tilde{\wp_\jmath})^{\ell_\jmath} =$$

$$\left\langle \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\mathfrak{T}_{\tilde{\wp_\jmath}}^{\ell_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\mathfrak{T}_{\tilde{\wp_\jmath}}^2)\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\mathfrak{T}_{\tilde{\wp_\jmath}}^{2\ell_\jmath}}}, \sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph_{\tilde{\wp_\jmath}}^2\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\aleph_{\tilde{\wp_\jmath}}^2)^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph_{\tilde{\wp_\jmath}}^2\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\aleph_{\tilde{\wp_\jmath}}^2)^{\ell_\jmath}}} \right\rangle.$$

Theorem 5.2 (Idempotency property)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). If $\tilde{\wp_\jmath} = \tilde{\wp}$ for all $\jmath = 1, 2, ..., \hbar$ then $PyFHWG(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \tilde{\wp}$.

Theorem 5.3 (Boundness property)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$) with the total order relation $\leq$ defined on it. If $\tilde{\wp}^+ = \langle \mathfrak{T}_{\tilde{\wp}^+}, \aleph_{\tilde{\wp}^+} \rangle = \max_\jmath\{\tilde{\wp_\jmath}\} = \max_\jmath\{\langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle\}$ and $\tilde{\wp}^- = \langle \mathfrak{T}_{\tilde{\wp}^-}, \aleph_{\tilde{\wp}^-} \rangle = \min_\jmath\{\tilde{\wp_\jmath}\} = \min_\jmath\{\langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle\}$ then $PyFHWG_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ is bounded as below
$\tilde{\wp}^- \leq PyFHWG_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) \leq \tilde{\wp}^+$.

Theorem 5.4 (Monotonicity property)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} : \jmath = 1, 2, ..., \hbar\}$ and $\mathfrak{P}' = \{\tilde{\wp_\jmath}' : \jmath = 1, 2, ..., \hbar\}$ be the two sets of PyFNs. If $\tilde{\wp_\jmath} \leq \tilde{\wp_\jmath}'$ then $PyFHWG_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) \leq PyFHWG_\ell(\tilde{\wp_1}', \tilde{\wp_2}', ..., \tilde{\wp_\hbar}')$.

5.2 Pythagorean Fuzzy Hamacher Ordered Weighted Geometric Operator

Definition 5.2 (Pythagorean fuzzy Hamacher ordered weighted geometric (PyFHOWG) operator)
Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFOWG operator is a mapping $PyFHOWG_\ell : \mathfrak{P}^\hbar \to \mathfrak{P}$ which is defined below.

$$PyFHOWG_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \bigotimes_{\jmath=1}^{\hbar}(\tilde{\wp}_{\sigma(\jmath)}^{\ell_\jmath}),$$

where $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ is a weight vector such that $\ell_\jmath \in [0, 1], \jmath = 1, 2, ..., \hbar$ and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1$ and $(\sigma(1), \sigma(2), ..., \sigma(\hbar))$ is a permutation of (1,2,...,$\hbar$) such that $\tilde{\wp}_{\sigma(\jmath-1)} \geq \tilde{\wp}_{\sigma(\jmath)}$ for all $\jmath = 2, 3, ..., \hbar$.
That is $PyFHOWG_\ell(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \tilde{\wp}_{\sigma(1)}^{\ell_1} \otimes \tilde{\wp}_{\sigma(2)}^{\ell_2} \otimes ... \otimes \tilde{\wp}_{\sigma(\hbar)}^{\ell_\hbar}$.

Theorem 5.5 Let $\mathfrak{P} = \{\tilde{\wp_\jmath} = \langle \mathfrak{T}_{\tilde{\wp_\jmath}}, \aleph_{\tilde{\wp_\jmath}} \rangle : \jmath = 1, 2, ..., \hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$).
Then

$$PyFHOWG_{\ell}(\tilde{\wp_1}, \tilde{\wp_2}, ..., \tilde{\wp_\hbar}) = \tilde{\wp}_{\sigma(1)}^{\ell_1} \otimes \tilde{\wp}_{\sigma(2)}^{\ell_2} \otimes ... \otimes \tilde{\wp}_{\sigma(\hbar)}^{\ell_\hbar} =$$

$$\left\langle \frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\mathfrak{T}_{\tilde{\wp}_{\sigma(\jmath)}}^{\ell_\jmath}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\mathfrak{T}_{\tilde{\wp}_{\sigma(\jmath)}}^2)\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\mathfrak{T}_{\tilde{\wp}_{\sigma(\jmath)}}^{2\ell_\jmath}}}, \right.$$

$$\sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph^2_{\tilde{\wp}_{\sigma(\jmath)}}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\aleph^2_{\tilde{\wp}_{\sigma(\jmath)}})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph^2_{\tilde{\wp}_{\sigma(\jmath)}}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\aleph^2_{\tilde{\wp}_{\sigma(\jmath)}})^{\ell_\jmath}}}\Bigg\rangle.$$

Theorem 5.6 (Idempotency property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). If $\tilde{\wp}_\jmath=\tilde{\wp}$ for all $\jmath=1,2,...,\hbar$ then $PyFHOWG_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\tilde{\wp}$.

Theorem 5.7 (Boundness property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$) with the total order relation $\leq$ defined on it. If $\tilde{\wp}^+=\langle\mathfrak{T}_{\tilde{\wp}^+},\aleph_{\tilde{\wp}^+}\rangle=\max_\jmath\{\tilde{\wp}_\jmath\}=\max_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ and $\tilde{\wp}^-=\langle\mathfrak{T}_{\tilde{\wp}^-},\aleph_{\tilde{\wp}^-}\rangle=\min_\jmath\{\tilde{\wp}_\jmath\}=\min_\jmath\{\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\}$ then $PyFHOWG_\ell\colon\mathfrak{P}^\hbar\to\mathfrak{P}$ is bounded as below:
$\tilde{\wp}^-\leq PyFHOWG_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)\leq\tilde{\wp}^+$.

Theorem 5.8 (Monotonicity property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath\colon \jmath=1,2,...,\hbar\}$ and $\mathfrak{P}'=\{\tilde{\wp}_\jmath{}'\colon \jmath=1,2,...,\hbar\}$ be the two sets of PyFNs. If $\tilde{\wp}_\jmath\leq\tilde{\wp}_\jmath{}'$ then $PyFHOWG_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)\leq PyFHOWG_\ell(\tilde{\wp}_1{}',\tilde{\wp}_2{}',...,\tilde{\wp}_\hbar{}')$.

Theorem 5.9 (Commutativity property)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath\colon \jmath=1,2,...,\hbar\}$ and $\mathfrak{P}'=\{\tilde{\wp}_\jmath{}'\colon \jmath=1,2,...,\hbar\}$ be the two sets of PyFNs. If $(\tilde{\wp}_1{}',\tilde{\wp}_2{}',...,\tilde{\wp}_\hbar{}')$ be any permutation of $(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)$ then $PyFHOWG_\ell(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=PyFHOWG_\ell(\tilde{\wp}_1{}',\tilde{\wp}_2{}',...,\tilde{\wp}_\hbar{}')$.

5.3 Pythagorean Fuzzy Hamacher Hybrid Geometric Operator

Definition 5.3 (Pythagorean fuzzy Hamacher hybrid geometric (PyFHHG) operator)
Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$). PyFHHG operator is a mapping $PyFHHG_{\ell,\Omega}\colon\mathfrak{P}^\hbar\to\mathfrak{P}$ which is defined below:

$$PyFHHG_{\ell,\Omega}(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=\bigotimes_{\jmath=1}^{\hbar}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath},$$

where $\ell=(\ell_1,\ell_2,...,\ell_\hbar)^T$ is a weight vector such that $\ell_\jmath\in[0,1],\jmath=1,2,...,\hbar$ and $\sum_{\jmath=1}^{\hbar}\ell_\jmath=1$ and $\tilde{\wp}^*_\jmath=(\tilde{\wp}_\jmath)^{\hbar\Omega_\jmath}$ and $(\sigma(1),\sigma(2),...,\sigma(\hbar))$ is a permutation of $(1,2,...,\hbar)$ such that $\tilde{\wp}^*_{\sigma(\jmath-1)}\geq\tilde{\wp}^*_{\sigma(\jmath)}$ for all $\jmath=2,3,...,\hbar$ and $\Omega=(\Omega_1,\Omega_2,...,\Omega_\hbar)^T$ is an associated weight vector such that $\Omega_\jmath\in[0,1]$ for all $\jmath=1,2,3,...,\hbar$ and $\sum_{\jmath=1}^{\hbar}\Omega_\jmath=1$.

Theorem 5.10 Let $\mathfrak{P}=\{\tilde{\wp}_\jmath=\langle\mathfrak{T}_{\tilde{\wp}_\jmath},\aleph_{\tilde{\wp}_\jmath}\rangle\colon \jmath=1,2,...,\hbar\}$ be the set of PyFNs in PyFN($\mathfrak{U}$).Then

$$PyFHHG_{\ell,\Omega}(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=$$
$$\Bigg\langle\frac{\sqrt{\kappa}\prod_{\jmath=1}^{\hbar}\mathfrak{T}^{\ell_\jmath}_{\tilde{\wp}^*_{\sigma(\jmath)}}}{\sqrt{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)(1-\mathfrak{T}^2_{\tilde{\wp}^*_{\sigma(\jmath)}})\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}\mathfrak{T}^{2\ell_\jmath}_{\tilde{\wp}^*_{\sigma(\jmath)}}}},$$
$$\sqrt{\frac{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph^2_{\tilde{\wp}^*_{\sigma(\jmath)}}\}^{\ell_\jmath}-\prod_{\jmath=1}^{\hbar}(1-\aleph^2_{\tilde{\wp}^*_{\sigma(\jmath)}})^{\ell_\jmath}}{\prod_{\jmath=1}^{\hbar}\{1+(\kappa-1)\aleph^2_{\tilde{\wp}^*_{\sigma(\jmath)}}\}^{\ell_\jmath}+(\kappa-1)\prod_{\jmath=1}^{\hbar}(1-\aleph^2_{\tilde{\wp}^*_{\sigma(\jmath)}})^{\ell_\jmath}}}\Bigg\rangle.$$

Theorem 5.11 If $\ell=(\frac{1}{\hbar},\frac{1}{\hbar},...,\frac{1}{\hbar})^T$ then $PyFHHG_{\ell,\Omega}(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)=PyFHWG_\Omega(\tilde{\wp}_1,\tilde{\wp}_2,...,\tilde{\wp}_\hbar)$.

Proof. We will prove it by Mathematical induction.
Let $\ell=(\ell_1,\ell_2)^T=(\frac{1}{2},\frac{1}{2})^T$ and $\Omega=(\Omega_1,\Omega_2)^T$ be an associated weight vector.
Therefore $\tilde{\wp}^*_\jmath=(\tilde{\wp}_\jmath)^{2\Omega_\jmath}$ for $\jmath=1,2$.
We suppose, without loss of generality, that $\tilde{\wp}^*_2\geq\tilde{\wp}^*_1$ $i.e.$ $\tilde{\wp}^*_{\sigma(1)}\geq\tilde{\wp}^*_{\sigma(2)}$.

Now, $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2}) = \bigotimes_{\jmath=1}^{2}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath} = (\tilde{\wp}^*_{\sigma(1)})^{\ell_1} \otimes (\tilde{\wp}^*_{\sigma(2)})^{\ell_2}$

$= (\tilde{\wp}^*_2)^{\ell_1} \otimes (\tilde{\wp}^*_1)^{\ell_2} = \{(\tilde{\wp_2})^{2\Omega_2}\}^{\ell_1} \otimes \{(\tilde{\wp_1})^{2\Omega_1}\}^{\ell_2} = \bigotimes_{\jmath=1}^{2}(\tilde{\wp_\jmath})^{\Omega_\jmath} = PyFHWG_\Omega(\tilde{\wp_1},\tilde{\wp_2})$.

Thus the theorem is true for $\hbar = 2$.

We assume that the theorem is true for $\hbar = \varsigma \in \mathbf{N}$ *i.e.*, $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})$ $= PyFHWG_\Omega(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})$ *i.e.*, $(\tilde{\wp}^*_{\sigma(1)})^{\ell_1} \otimes (\tilde{\wp}^*_{\sigma(2)})^{\ell_2}\otimes,...,\otimes(\tilde{\wp}^*_{\sigma(\varsigma)}) = (\tilde{\wp_1})^{\Omega_1} \otimes (\tilde{\wp_2})^{\Omega_2}\otimes,...,\otimes(\tilde{\wp_\varsigma})^{\Omega_\varsigma}$ and in this case $\ell = (\ell_1,\ell_2,...,\ell_\varsigma)^T = (\frac{1}{\varsigma},\frac{1}{\varsigma},...,\frac{1}{\varsigma})^T$ and $\tilde{\wp}^*_\jmath = (\tilde{\wp_\jmath})^{\varsigma\Omega_\jmath}$,

$\jmath = 1,2,...,\varsigma$.

Now,$PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1}) =$

$\underbrace{(\tilde{\wp}^*_{\sigma(1)})^{\ell_1} \otimes (\tilde{\wp}^*_{\sigma(2)})^{\ell_2}\otimes,...,\otimes(\tilde{\wp}^*_{\sigma(\varsigma)})^{\ell_\varsigma}}_{\varsigma\text{terms}} \otimes(\tilde{\wp}^*_{\sigma(\varsigma+1)})^{\ell_{\varsigma+1}}$.

Now, it is obvious that $\tilde{\wp}^*_{\sigma(\varsigma+1)} = \tilde{\wp}^*_{\varsigma+1}$ and $\ell_{\varsigma+1} = \frac{1}{\varsigma+1}$.

Thus $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1}) = \underbrace{(\tilde{\wp_1})^{\Omega_1} \otimes (\tilde{\wp_2})^{\Omega_2}\otimes,...,\otimes(\tilde{\wp_\varsigma})^{\Omega_\varsigma}}_{\varsigma\text{terms}} \otimes(\tilde{\wp}_{\varsigma+1})^{\Omega_{\varsigma+1}}$

$= \bigotimes_{\jmath=1}^{\varsigma+1}(\tilde{\wp_\jmath})^{\Omega_\jmath} = PyFHWG_\Omega(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp}_{\varsigma+1})$.

Thus the theorem is true for $\hbar = \varsigma + 1$ when it is assumed to be true for $\hbar = \varsigma$.

Hence, the theorem is valid for all $\hbar \in \mathbf{N}$ by Mathematical induction.

Theorem 5.12 If $\Omega = (\frac{1}{\hbar},\frac{1}{\hbar},...,\frac{1}{\hbar})^T$ then $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar}) = PyFHOWG_\ell(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\hbar})$

Proof. Let $\Omega = (\Omega_1,\Omega_2) = (\frac{1}{2},\frac{1}{2})^T$ and $\ell = (\ell_1,\ell_2)^T$ be weight vectors.

Now, $\tilde{\wp}^*_\jmath = (\tilde{\wp_\jmath})^{2\Omega_\jmath}$, which becomes $\tilde{\wp}^*_\jmath = \tilde{\wp_\jmath}$ for all $\jmath = 1,2$.

We suppose, without loss of generality, that $\tilde{\wp}^*_2 \geq \tilde{\wp}^*_1$ *i.e.*, $\tilde{\wp}^*_{\sigma(1)} \geq \tilde{\wp}^*_{\sigma(2)}$.

Now, $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2}) = \bigotimes_{\jmath=1}^{2}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath} = (\tilde{\wp}^*_{\sigma(1)})^{\ell_1} \otimes (\tilde{\wp}^*_{\sigma(2)})^{\ell_2} = \bigotimes_{\jmath=1}^{2}(\tilde{\wp}_{\sigma(\jmath)})^{\ell_\jmath}$

$= PyFHOWG_\ell(\tilde{\wp_1},\tilde{\wp_2})$.Thus the theorem is true for $\hbar = 2$.

We suppose that the theorem is true for $\hbar = \varsigma \in \mathbf{N}$.Then we can write that

$PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma}) = PyFHOWG_\ell(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma})$ *i.e.*, $\bigotimes_{\jmath=1}^{\varsigma}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath} = \bigotimes_{\jmath=1}^{\varsigma}(\tilde{\wp}_{\sigma(\jmath)})^{\ell_\jmath}$ and in this case

$\Omega = (\Omega_1,\Omega_2,...,\Omega_\varsigma)^T = (\frac{1}{\varsigma},\frac{1}{\varsigma},...,\frac{1}{\varsigma})^T$ and $\tilde{\wp}^*_\jmath = (\tilde{\wp_\jmath})^{\varsigma\Omega_\jmath}$,

$\jmath = 1,2,...,\varsigma$.

Now, $PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1}) = \bigotimes_{\jmath=1}^{\varsigma+1}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath} = \bigotimes_{\jmath=1}^{\varsigma}(\tilde{\wp}^*_{\sigma(\jmath)})^{\ell_\jmath} \otimes (\tilde{\wp}^*_{\sigma(\varsigma+1)})^{\ell_{\varsigma+1}}$.

In this case, we have $\tilde{\wp}^*_\jmath = (\tilde{\wp_\jmath})^{(\varsigma+1)\Omega_\jmath}$, $\jmath = 1,2,...,(\varsigma+1)$. Obviously we can write that $\tilde{\wp}^*_{\sigma(\varsigma+1)} = \tilde{\wp}_{\sigma(\varsigma+1)}$ and $\Omega_{\varsigma+1} = \frac{1}{\varsigma+1}$.

$PyFHHG_{\ell,\Omega}(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1}) = \bigotimes_{\jmath=1}^{\varsigma}(\tilde{\wp}_{\sigma(\jmath)})^{\ell_\jmath} \otimes (\tilde{\wp}_{\sigma(\varsigma+1)})^{\ell_{\varsigma+1}} = \bigotimes_{\jmath=1}^{\varsigma+1}(\tilde{\wp}_{\sigma(\jmath)})^{\ell_\jmath}$

$= PyFHOWG_\ell(\tilde{\wp_1},\tilde{\wp_2},...,\tilde{\wp_\varsigma},\tilde{\wp}_{\varsigma+1})$.

Thus the theorem is valid for $\hbar = \varsigma + 1$ when it is supposed to be true for $\hbar = \varsigma$. Hence, by Mathematical induction, the theorem is valid for all $\hbar \in\mathbf{N}$.

6 Algorithm for Multi-Attribute Decision Making Using PyFI

This section proposes a new approach to the decision-making problem using Pythagorean fuzzy information. In this approach, experts provide their information in the form of PyFSs.

Let $A = \{A_1, A_2, ..., A_\varsigma\}$ be the set of ς distinct alternatives and $B = \{B_1, B_2, ..., B_\hbar\}$ be the set of $\hbar$ distinct attributes. Let a specific number of experts (decision makers) give their decisions towards different alternatives in PyFE based on predefined attributes. Let $\tilde{P_{ij}} = \{\langle \Im_{\tilde{P_{ij}}}, \aleph_{\tilde{P_{ij}}}\rangle : \imath = 1,2,...,\varsigma$ and $\jmath = 1,2,...,\hbar\}$ be the PyFI given by the experts in aggregated form towards the $\imath$-th alternative to the basis of $\jmath$-th attribute. In this way, we can form a matrix called decision-making matrix $D = [\tilde{P_{ij}}]_{\varsigma\times\hbar}$.

Our target is to aggregate the PyFI obtained in the decision-making matrix corresponding to each alternative and find the best alternative. Different attributes are assigned different weights during the evaluation of aggregated values corresponding to each alternative to fulfil the expected requirements of the decision-makers. Let $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ be a weight vector such that $\ell_\imath$ be the weight assigned to the $\jmath$-th attribute and $\sum_{\jmath=1}^{\hbar} \ell_\jmath = 1, \ell_\jmath \in [0,1]$.

The algorithm of the solution to the multi-attribute decision-making problem is as follows:

Step-1: In this step, we get collective information in the decision-making matrix form in PyFE. Then we aggregate that information using the proposed Pythagorean fuzzy Hamacher weighted averaging (PyFHWA) operator and Pythagorean fuzzy Hamacher weighted geometric (PyFHWG) operator with the associated weight vector $\ell = (\ell_1, \ell_2, ..., \ell_\hbar)^T$ such that $\sum_{\jmath=1}^{\hbar} \ell_\imath = 1, \ell_\jmath \in [0,1]$.

Step-2: We find the score values and accuracy values (if needed) of the aggregated PyFN concerning each alternative.

Step-3: Construction of rank of the alternatives $B_\imath(\imath = 1, 2, ..., \hbar)$ based on score values and accuracy values (if needed) and selection of the best alternative occupying maximum rank (See Figure 2).

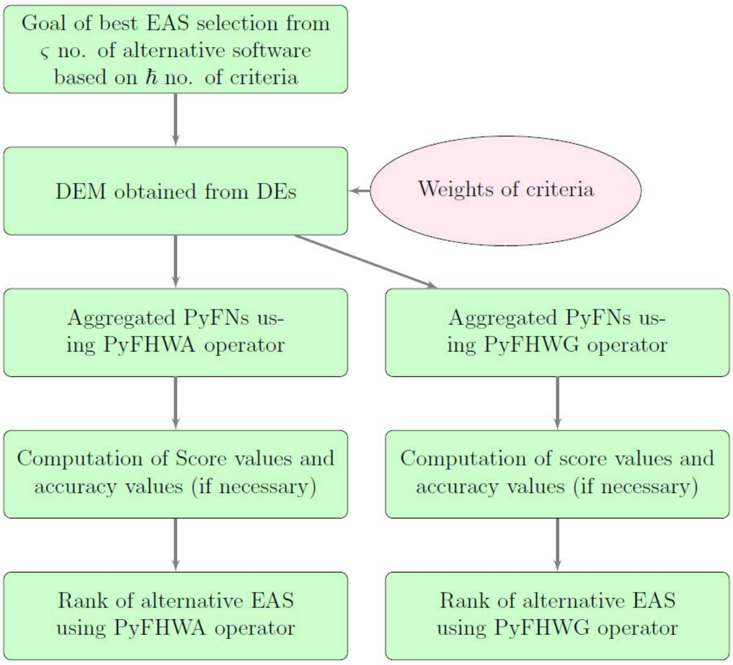

Figure 2. Diagram of the present model.

6.1 Numerical Example

Enterprise Application Software (EAS) is also known as Enterprise Software (ES). Let an organization seeking the best EAS among the four reputed EAS companies, namely, A_1, A_2, A_3, A_4 which are assumed as alternatives and B_1, B_2, B_3, B_4, B_5 be the five predetermined criteria (attributes) based on which the best alternative is to be chosen. The EASs mostly used in large business organizations comprising of different rolls and activities. They normally includes sales department, information technology sector, finance sector, juridical part and public dealings. Let B_1, B_2, B_3, B_4, B_5 represent the attributes Credibility, Agility, User-friendliness, Compatibility, and Less market price, respectively. The organization determines the weight $\ell_\jmath(\jmath = 1, 2, 3, 4, 5)$ corresponding to the attribute $B_\jmath(\jmath = 1, 2, 3, 4, 5)$ so that their importance on specific attribute in best alternative selection be fulfilled.

The DEM $D = \left[\tilde{P}_{ij}\right]_{4\times5}$ containing the PyFI provided by the specific number of experts towards the different alternatives is given in Table 2.

Case-1:

Step-1: Now we use the PyFHWA operator to determine the aggregated performance of $A_\imath(\imath = 1, 2, 3, 4)$ based on the attributes $B_\jmath(\jmath = 1, 2, 3, 4, 5)$. These performances are shown in Table 3 with respect to the weight vector $\ell = (0.25, 0.15, 0.10, 0.35, 0.15)^T$ and parameter $\kappa = 1$.

From Table 3, we can find the score values corresponding to each aggregated PyFNs under the PyFHWA operator as follows:

Table 2. Pythagorean fuzzy DEM in tabular form

	B_1	B_2	B_3	B_4	B_5
A_1	$\langle 0.7, 0.4 \rangle$	$\langle 0.4, 0.5 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.3, 0.4 \rangle$	$\langle 0.6, 0.4 \rangle$
A_2	$\langle 0.7, 0.3 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.7, 0.4 \rangle$
A_3	$\langle 0.3, 0.2 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.3, 0.6 \rangle$	$\langle 0.6, 0.5 \rangle$	$\langle 0.6, 0.2 \rangle$
A_4	$\langle 0.7, 0.3 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.5, 0.4 \rangle$

Table 3. Aggregated PyFNs under PyFHWA operator for $\kappa = 1$

	Aggregated value(PyFN)
A_1	$\langle 0.5363, 0.4019 \rangle$
A_2	$\langle 0.6547, 0.4610 \rangle$
A_3	$\langle 0.5480, 0.3076 \rangle$
A_4	$\langle 0.6363, 0.3366 \rangle$

$Sc(A_1) = 0.1261$, $Sc(A_2) = 0.2162$, $Sc(A_3) = 0.2057$, $Sc(A_4) = 0.2916$. Similarly, we can find the same for the other parameter values, $\kappa = 2, 3, 4, 5, 6, 7, 8, 9, 10$.

Step-2 and 3: The score values of the aggregated PyFNs and their ranks are shown in Table 4.

Case 2:

Step 1: Now we use the PyFHWG operator to determine the aggregated performance of $A_\imath (\imath = 1, 2, 3, 4)$ based on the attributes $B_\jmath (\jmath = 1, 2, 3, 4, 5)$. These performances are shown in Table 5 concerning the weight vector $\ell = (0.25, 0.15, 0.10, 0.35, 0.15)^T$ and parameter $\kappa = 1$. From Table 5, we can find the score values corresponding to each aggregated PyFNs and which are as follows:

$Sc(A_1) = 0.0444$, $Sc(A_2) = 0.0622$, $Sc(A_3) = 0.0776$, $Sc(A_4) = 0.2557$. Similarly, we can find the same for other κ values.

Step 2 and 3: The score values of aggregated PyFNs under the PyFHWG operator and their ranks are shown in Table 6.

Table 4. Ranks of aggregated PyFNs under PyFHWA operator

κ	$Sc(A_1)$	$Sc(A_2)$	$Sc(A_3)$	$Sc(A_4)$	Ranking
1	0.1261	0.2162	0.2057	0.2916	$A_1 < A_3 < A_2 < A_4$
2	0.1147	0.2023	0.1914	0.2866	$A_1 < A_3 < A_2 < A_4$
3	0.1073	0.1955	0.1875	0.2836	$A_1 < A_3 < A_2 < A_4$
4	0.1021	0.1914	0.1826	0.2817	$A_1 < A_3 < A_2 < A_4$
5	0.0981	0.1827	0.1788	0.2803	$A_1 < A_3 < A_2 < A_4$
6	0.0949	0.1868	0.1759	0.2793	$A_1 < A_3 < A_2 < A_4$
7	0.0924	0.1853	0.1735	0.2785	$A_1 < A_3 < A_2 < A_4$
8	0.0903	0.1842	0.1715	0.2779	$A_1 < A_3 < A_2 < A_4$
9	0.0885	0.1833	0.1698	0.2774	$A_1 < A_3 < A_2 < A_4$
10	0.0870	0.1826	0.1684	0.2770	$A_1 < A_3 < A_2 < A_4$

6.2 Analysis of Dependency on the Parameter κ in MADM Result

To describe the effect of the parameter κ in MADM result, we take ten different values of κ in PyFHWA and PyFHWG operators, and then we calculate the score values of aggregated PyFNs using two different operators corresponding to four alternatives and after that ranks of the alternatives are calculated. Scores and ranks are shown in Table 4 and Table 6, respectively.

From Table 4, it is clear that the ranks remain the same for different values of κ $i.e.$, the result is not affected by the values of κ, and for all κ the rank of the alternatives is $A_1 < A_3 < A_2 < A_4$, $i.e.$, the best alternative deduced is A_4 for the MADM problem based on PyFHWA operator.

From Table 6, it is noteworthy that the rank differs after the change of κ values.

For $1 \leq \kappa \leq 2$, the rank of the alternatives is $A_1 < A_2 < A_3 < A_4$ and consequently, the best preferable alternative is A_4. In this case, κ values do not affect ranks as well as the result of the MADM problem.

Now for $3 \leq \kappa \leq 10$ the rank of alternatives is $A_1 < A_3 < A_2 < A_4$ and consequently the best preferable alternative is A_4. Hence the best alternative is A_4 under the PyFHWG operator. In this case, κ values affect the

Table 5. Aggregated PyFNs under PyFHWG operator for $\kappa = 1$

	Aggregated value(PyFN)
A_1	$\langle 0.4604, 0.4093 \rangle$
A_2	$\langle 0.6481, 0.5982 \rangle$
A_3	$\langle 0.4818, 0.3930 \rangle$
A_4	$\langle 0.6119, 0.3445 \rangle$

ranking of aggregated PyFNs, but despite this, the results remain the same.

Table 6. Ranks of aggregated PyFNs under PyFHWG operator

κ	$Sc(A_1)$	$Sc(A_2)$	$Sc(A_3)$	$Sc(A_4)$	Rank
1	0.0444	0.0622	0.0776	0.2557	$A_1 < A_2 < A_3 < A_4$
2	0.0522	0.0891	0.0912	0.2607	$A_1 < A_2 < A_3 < A_4$
3	0.0556	0.1049	0.0990	0.2632	$A_1 < A_3 < A_2 < A_4$
4	0.0577	0.1154	0.1044	0.2647	$A_1 < A_3 < A_2 < A_4$
5	0.0591	0.1231	0.1085	0.2657	$A_1 < A_3 < A_2 < A_4$
6	0.0600	0.1290	0.1118	0.2664	$A_1 < A_3 < A_2 < A_4$
7	0.0608	0.1336	0.1145	0.2670	$A_1 < A_3 < A_2 < A_4$
8	0.0614	0.1373	0.1169	0.2675	$A_1 < A_3 < A_2 < A_4$
9	0.0691	0.1404	0.1188	0.2679	$A_1 < A_3 < A_2 < A_4$
10	0.0623	0.1431	0.1206	0.2682	$A_1 < A_3 < A_2 < A_4$

7 Conclusion

In this proposed article, we have discussed solving procedure of MADM issues using different averaging (PyFHWA, PyFHOWA, PyFHHA) operators and different geometric (PyFHWG, PyFHOWA, PyFHHA) operators. Though the article deals with the averaging and geometric operators separately, the algorithm and an example have been provided based on the PyFHWA and PyFHWG operators. This article considers the selection procedure of enterprise application software in the framework of multi-attribute decision-making problems. The four alternative application software are considered in the primary stage. The five criteria are considered for choosing the best software. The DEM proposed by the DEs is in the PyFE. Consequently, each piece of information for each alternative software corresponding to each criterion is PyFN. The aggregation is performed using the PyFHWA and PyFHWG operators separately. The score values of the aggregated PyFNs show that A_4 possesses the highest score value 0.2916 under the PyFHWA operator and 0.2557 under the PyFHWG operator for $\kappa = 1$. Therefore A_4 is the best EAS followed by A_2, A_3 and A_1 under the PyFHWA operator, and A_4 is also the best EAS under the PyFHWG operator followed by A_3, A_2 and A_1. The Table 4 shows that the parameter κ has no such remarkable effect in ranking orders of alternatives because if we vary κ values 1 to 10 although the score values change the ranking order of alternatives remain unaltered in either case. But, κ has considerable influence in ranking order of alternatives in Table 3 although the highest and lowest EAS remain the same.

The present method is applied in EAS selection, considering four alternatives and five criteria. The proposed method can also be applied in multi-criteria group decision-making techniques where the information might be fuzzy or its extended forms like intuitionistic, Pythagorean, q-rung orthopair, neutrosophic, triangular, trapezoidal fuzzy environment. Experts' DEMs can be converted into a single DEM using Pythagorean fuzzy Hamacher weighted or ordered weighted or hybrid aggregation operators. These extended methods can be utilized in selecting efficient waste-to-energy technology, compatible logistics supplier selection, etcetera.

The present method considers the criteria' weights vector as $(0.25, 0.15, 0.10, 0.35, 0.15)^T$, and it is arbitrarily chosen. No method is applied for determining this weight vector. Also, the DEM considered in this article is assumed as aggregated DEM obtained after the aggregation of DEMs of the DEs. The DEMs are not separately proposed by the DEs. Experts' weights are not mentioned explicitly in this article, and they are treated as identical for each DE. Normally, The weights are distributed amongst the DEs based on their expertise in different fields.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] D. Liang and Z. Xu, "The new extension of topsis method for multiple criteria decision making with hesitant pythagorean fuzzy sets," *Appl. Soft Comput.*, vol. 60, pp. 167–179, 2017. https://doi.org/10.1016/j.asoc.2017.06.034

[2] X. Zhang, "A novel approach based on similarity measure for pythagorean fuzzy multiple criteria group decision making," *Inter. J. Int. Syst.*, vol. 31, no. 6, pp. 593–611, 2016. https://doi.org/10.1002/int.21796

[3] H. Garg, "Some methods for strategic decision-making problems with immediate probabilities in pythagorean fuzzy environment," *Inter. J. Int. Syst.*, vol. 33, no. 4, pp. 687–712, 2018. https://doi.org/10.1002/int.21949

[4] C. Yu, Y. Shao, K. Wang, and L. Zhang, "A group decision making sustainable supplier selection approach using extended topsis under interval-valued pythagorean fuzzy environment," *Expert Syst. with Appl.*, vol. 121, pp. 1–17, 2019. https://doi.org/10.1016/j.eswa.2018.12.010

[5] X. Zhang, "Multicriteria pythagorean fuzzy decision analysis: A hierarchical qualiflex approach with the closeness index-based ranking methods," *Infor. Sci.*, vol. 330, pp. 104–124, 2016. https://doi.org/10.1016/j.ins.2015.10.012

[6] P. Ren, Z. Xu, and X. Gou, "Pythagorean fuzzy todim approach to multi-criteria decision making," *Appl. Soft Comput.*, vol. 42, pp. 246–259, 2016. https://doi.org/10.1016/j.asoc.2015.12.020

[7] M. Sajjad Ali Khan, S. Abdullah, M. Yousaf Ali, I. Hussain, and M. Farooq, "Extension of topsis method base on choquet integral under interval-valued pythagorean fuzzy environment," *J. Int. & Fuzzy Syst.*, vol. 34, no. 1, pp. 267–282, 2018. https://doi.org/10.3233/JIFS-171164

[8] J. C. Fodor and M. Roubens, *Fuzzy preference modelling and multicriteria decision support*. Springer Science & Business Media, 1994, vol. 14.

[9] S. Greco, B. Matarazzo, and R. Slowinski, "Rough sets theory for multicriteria decision analysis," *Eur. J.Oper. Res.*, vol. 129, no. 1, pp. 1–47, 2001. https://doi.org/10.1016/S0377-2217(00)00167-3

[10] W. Ho, X. Xu, and P. K. Dey, "Multi-criteria decision making approaches for supplier evaluation and selection: A literature review," *Eur. J.Oper. Res.*, vol. 202, no. 1, pp. 16–24, 2010. https://doi.org/10.1016/j.ejor.2009.05.009

[11] C. Kahraman, U. Cebeci, and Z. Ulukan, "Multi-criteria supplier selection using fuzzy ahp," *Logistics Info. Manage.*, vol. 16, no. 6, pp. 382–394, 2003. https://doi.org/10.1108/09576050310503367

[12] J. Malczewski, "Gis-based multicriteria decision analysis: A survey of the literature," *Inter. J. Geogr. Info. Sci.*, vol. 20, no. 7, pp. 703–726, 2006. https://doi.org/10.1080/13658810600661508

[13] A. Mardani, E. K. Zavadskas, Z. Khalifah, N. Zakuan, A. Jusoh, K. M. Nor, and M. Khoshnoudi, "A review of multi-criteria decision-making applications to solve energy management problems: Two decades from 1995 to 2015," *Renewable Sust. Energ. R.*, vol. 71, pp. 216–256, 2017. https://doi.org/10.1016/j.rser.2016.12.053

[14] S. W. Myint and L. Wang, "Multicriteria decision approach for land use land cover change using markov chain analysis and a cellular automata approach," *Canadian J. Remote Sensing*, vol. 32, no. 6, pp. 390–404, 2006. https://doi.org/10.5589/m06-032

[15] S. D. Pohekar and M. Ramachandran, "Application of multi-criteria decision making to sustainable energy planning—a review," *Renewable Sust. Energ. R.*, vol. 8, no. 4, pp. 365–381, 2004. https://doi.org/10.1016/j.rser.2003.12.007

[16] H. Rey-Valette, S. Damart, and S. Roussel, "A multicriteria participation-based methodology for selecting sustainable development indicators: An incentive tool for concerted decision making beyond the diagnosis framework," *Inter. J. Sust. Dev.*, vol. 10, no. 1-2, pp. 122–138, 2007.

[17] A. Silvestri, F. De Felice, and A. Petrillo, "Multi-criteria risk analysis to improve safety in manufacturing systems," *Inter. J. Proc. Res.*, vol. 50, no. 17, pp. 4806–4821, 2012. https://doi.org/10.1080/00207543.2012.657968

[18] K. T. Atanassov, "New operations defined over the intuitionistic fuzzy sets," *Fuzzy Sets & Syst.*, vol. 61, no. 2, pp. 137–142, 1994.

[19] L. A. Zadeh, "Fuzzy sets," *Info. & Control*, vol. 8, no. 3, pp. 338–353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[20] R. R. Yager, "Pythagorean fuzzy subsets," pp. 57–61, 2013. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[21] C. Jana, T. Senapati, and M. Pal, "Pythagorean fuzzy dombi aggregation operators and its applications in multiple attribute decision-making," *Inter. J. Int. Syst.*, vol. 34, no. 9, pp. 2019–2038, 2019. https://doi.org/10.1002/int.22125

[22] A. A. Khan, S. Ashraf, S. Abdullah, M. Qiyas, J. Luo, and S. U. Khan, "Pythagorean fuzzy dombi aggregation operators and their application in decision support system," *Symmetry*, vol. 11, no. 3, p. 383, 2019. https://doi.org/10.3390/sym11030383

[23] C. Jana, M. Pal, and J.-q. Wang, "Bipolar fuzzy dombi aggregation operators and its application in multiple-attribute decision-making process," *J. Ambient Int. & Humanized Comput.*, vol. 10, pp. 3533–3549, 2019. https://doi.org/10.1007/s12652-018-1076-9

[24] H. Garg, "A new generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making," *Inter. J. Int. Syst.*, vol. 31, no. 9, pp. 886–920, 2016. https://doi.org/10.1002/int.21809

[25] H. Garg, "Generalized pythagorean fuzzy geometric aggregation operators using einstein t-norm and t-conorm for multicriteria decision-making process," *Inter. J. Int. Syst.*, vol. 32, no. 6, pp. 597–630, 2017. https://doi.org/10.1002/int.21860

[26] H. Garg, "Generalised pythagorean fuzzy geometric interactive aggregation operators using einstein operations and their application to decision making," *J. Exp. & Th. Artif. Int.*, vol. 30, no. 6, pp. 763–794, 2018. https://doi.org/10.1080/0952813X.2018.1467497

[27] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, "Pythagorean fuzzy einstein weighted geometric aggregation operator and their application to multiple attribute group decision making," *J. Int. & Fuzzy Syst.*, vol. 33, no. 1, pp. 635–647, 2017. https://doi.org/10.3233/JIFS-16797

[28] H. Gao, "Pythagorean fuzzy hamacher prioritized aggregation operators in multiple attribute decision making," *J. Int. & Fuzzy Syst.*, vol. 35, no. 2, pp. 2229–2245, 2018. https://doi.org/10.3233/JIFS-172262

[29] H. Gao, G. Wei, and Y. Huang, "Dual hesitant bipolar fuzzy hamacher prioritized aggregation operators in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 11 508–11 522, 2017. https://doi.org/10.1109/ACCESS.2017.2784963

[30] J.-Y. Huang, "Intuitionistic fuzzy hamacher aggregation operators and their application to multiple attribute decision making," *J. Int. & Fuzzy Syst.*, vol. 27, no. 1, pp. 505–513, 2014. https://doi.org/10.3233/IFS-131019

[31] C. Jana, G. Muhiuddin, and M. Pal, "Multiple-attribute decision making problems based on svtnh methods," *J. Ambient Int. & Humanized Comput.*, vol. 11, pp. 3717–3733, 2020. https://doi.org/10.1007/s12652-019-01568-9

[32] C. Jana and M. Pal, "Assessment of enterprise performance based on picture fuzzy hamacher aggregation operators," *Symmetry*, vol. 11, no. 1, p. 75, 2019. https://doi.org/10.3390/sym11010075

[33] Y. Ju, W. Zhang, and S. Yang, "Some dual hesitant fuzzy hamacher aggregation operators and their applications to multiple attribute decision making," *J. Int. & Fuzzy Syst.*, vol. 27, no. 5, pp. 2481–2495, 2014. https://doi.org/10.3233/IFS-141222

[34] P. Liu, "Some hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 83–97, 2013. https://doi.org/10.1109/TFUZZ.2013.2248736

[35] P. Liu, Y. Chu, Y. Li, and Y. Chen, "Some generalized neutrosophic number hamacher aggregation operators and their application to group decision making," *Inter. J. Fuzzy Syst.*, vol. 16, no. 2, 2014.

[36] M. Lu, G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making," *IFS*, vol. 33, no. 2, pp. 1105–1117, 2017. https://doi.org/10.3233/JIFS-16554

[37] C. Tan, W. Yi, and X. Chen, "Hesitant fuzzy hamacher aggregation operators for multicriteria decision making," *Appl. Soft Comput.*, vol. 26, pp. 325–349, 2015. https://doi.org/10.1016/j.asoc.2014.10.007

[38] J. Tang and F. Meng, "Linguistic intuitionistic fuzzy hamacher aggregation operators and their application to group decision making," *Granul. Comput.*, vol. 4, pp. 109–124, 2019. https://doi.org/10.1007/s41066-018-0089-2

[39] X. Tang, C. Fu, D.-L. Xu, and S. Yang, "Analysis of fuzzy hamacher aggregation functions for uncertain multiple attribute decision making," *Info. Sci.*, vol. 387, pp. 19–33, 2017. https://doi.org/10.1016/j.ins.2016.12.045

[40] N. Waseem, M. Akram, and J. C. R. Alcantud, "Multi-attribute decision-making based on m-polar fuzzy hamacher aggregation operators," *Symmetry*, vol. 11, no. 12, p. 1498, 2019. https://doi.org/10.3390/sym11121498

[41] G. Wei, "Picture fuzzy hamacher aggregation operators and their application to multiple attribute decision making," *FI*, vol. 157, no. 3, pp. 271–320, 2018. https://doi.org/10.3233/FI-2018-1628

[42] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Bipolar fuzzy hamacher aggregation operators in multiple attribute decision making," *Int. J. Fuzzy Syst.*, vol. 20, pp. 1–12, 2018. https://doi.org/10.1007/s40815-017-0338-6

[43] G. Wei and M. Lu, "Dual hesitant pythagorean fuzzy hamacher aggregation operators in multiple attribute decision making," *Arch. Control Sci.*, vol. 27, no. 3, 2017. https://doi.org/10.1515/acsc-2017-0024

[44] G.-W. Wei, "Pythagorean fuzzy hamacher power aggregation operators in multiple attribute decision making," *FI*, vol. 166, no. 1, pp. 57–85, 2019. https://doi.org/10.3233/FI-2019-1794

[45] Q. Wu, P. Wu, L. Zhou, H. Chen, and X. Guan, "Some new hamacher aggregation operators under single-valued neutrosophic 2-tuple linguistic environment and their applications to multi-attribute group decision making," *Comput. & In. Eng.*, vol. 116, pp. 144–162, 2018. https://doi.org/10.1016/j.cie.2017.12.024

[46] L. Zhou, X. Zhao, and G. Wei, "Hesitant fuzzy hamacher aggregation operators and their application to multiple attribute decision making," *J. Int. & Fuzzy Syst.*, vol. 26, no. 6, pp. 2689–2699, 2014. https://doi.org/10.3233/IFS-130939

[47] A. Garrido, "Triangular norms, triangular conorms, and some related concepts," *Broad Res.in Artif. Int. & Neurosc.*, vol. 2, no. 1, pp. 59–62, 2011.

[48] E. P. Klement, R. Mesiar, and E. Pap, "Problems on triangular norms and related operators," *Fuzzy Sets and Syst.*, vol. 145, no. 3, pp. 471–479, 2004. https://doi.org/10.1016/S0165-0114(03)00303-8

[49] M. Z. Reformat and R. R. Yager, "Suggesting recommendations using pythagorean fuzzy sets illustrated using netflix movie data," pp. 546–556, 2014. https://doi.org/10.1007/978-3-319-08795-5_56

[50] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *Int. J. Intell. Syst.*, vol. 28, no. 5, pp. 436–452, 2013. https://doi.org/10.1002/int.21584

[51] M. M. Gupta and J. Qi, "Theory of t-norms and fuzzy inference methods," *Fuzzy Sets & Syst.*, vol. 40, no. 3, pp. 431–450, 1991.

[52] X. Zhang and Z. Xu, "Extension of topsis to multiple criteria decision making with pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 29, no. 12, pp. 1061–1078, 2014. https://doi.org/10.1002/int.21676

[53] X. Peng and Y. Yang, "Some results for pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 30, no. 11, pp. 1133–1160, 2015. https://doi.org/10.1002/int.21738

[54] T.-Y. Chen, "An interval-valued pythagorean fuzzy outranking method with a closeness-based assignment model for multiple criteria decision making," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 126–168, 2018. https://doi.org/10.1002/int.21943

[55] T.-Y. Chen, "Remoteness index-based pythagorean fuzzy vikor methods with a generalized distance measure for multiple criteria decision analysis," *Infor. Fusion*, vol. 41, pp. 129–150, 2018. https://doi.org/10.1016/j.inffus.2017.09.003

[56] S. Dick, R. R. Yager, and O. Yazdanbakhsh, "On pythagorean and complex fuzzy set operations," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 5, pp. 1009–1021, 2015. https://doi.org/10.1109/TFUZZ.2015.2500273

[57] R. R. Yager, "Pythagorean fuzzy subsets," pp. 57–61, 2013. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[58] X. Peng and H. Yuan, "Fundamental properties of pythagorean fuzzy aggregation operators," *FI*, vol. 147, no. 4, pp. 415–446, 2016. https://doi.org/10.3233/FI-2016-1415

[59] K. Rahman, A. Ali, M. Shakeel, M. A. Khan, and M. Ullah, "Pythagorean fuzzy weighted averaging aggregation operator and its application to decision making theory," *The Nucleus*, vol. 54, no. 3, pp. 190–196, 2017.

[60] K. Rahman, S. Abdullah, A. Ali, and F. Amin, "Pythagorean fuzzy ordered weighted averaging aggregation operator and their application to multiple attribute group decision-making," *EURO J. on Decision Processes*, vol. 8, pp. 61–77, 2020. https://doi.org/10.1007/s40070-020-00110-z

[61] S. Zeng, J. Chen, and X. Li, "A hybrid method for pythagorean fuzzy multiple-criteria decision making," *Int. J. Info. Tech. Dec. Mak.*, vol. 15, no. 02, pp. 403–422, 2016. https://doi.org/10.1142/S0219622016500012

[62] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, "Pythagorean fuzzy einstein weighted geometric aggregation operator and their application to multiple attribute group decision making," *IFS*, vol. 33, no. 1, pp. 635–647, 2017. https://doi.org/10.3233/JIFS-16797

[63] K. Rahman, S. Abdullah, M. A. Khan, and M. Shakeel, "Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making," *Inter. J. Comput. Sci. Info. Sc.*, vol. 14, no. 6, 2016.

[64] M. Lu, G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making," *IFS*, vol. 33, no. 2, pp. 1105–1117, 2017. https://doi.org/10.3233/JIFS-16554