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Interval-Valued Picture Fuzzy Uncertain Linguistic Dombi Operators and Their Application in Industrial Fund Selection

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Abstract: This study presents an advanced generalization of uncertain linguistic numbers (ULNs) and interval-valued intuitionistic uncertain linguistic numbers (IVIULNs) through the development of interval-valued picture fuzzy numbers (IVPFNs). Firstly, the IVPFUL weighted average and IVPFUL weighted geometric operators, denoted as IVPFULWA and IVPFULWG, have been introduced. Furthermore, the IVPFUL Dombi weighted average and geometric operators, represented by IVPFULDWA and IVPFULDWG, are also proposed in the same context. These operators are utilized to establish a multi-attribute decision-making (MADM) approach with IVPFUL data. Finally, the proposed methodology is applied to a mutual fund selection problem through a demonstrative example.

Keywords: Interval-valued picture fuzzy numbers(IVPFNs); Uncertain linguistic numbers (ULNs); IVPFULWA operator; IVPFULWG operator; Multi-attribute decision-making (MADM)

1 Introduction

The property of "refusal" is a crucial aspect that cannot be represented by traditional fuzzy sets (FSs) such as FSs [1] and IFS [2]. In response to this limitation, Coung [3, 4] introduced a novel concept called picture fuzzy sets (PFSs). Due to their significance, numerous researchers have endeavored to enhance the PFS concept and apply it to real-world decision-making processes. Efforts have been made to develop appropriate mathematical models that integrate various preferences of decision-makers into a collective preference for processing decision-making information. In this context, Wei [5] proposed picture fuzzy weighted average (PFWA), order (PFOWA), hybrid (PFHWA), geometric (PFWG), order weighted geometric (PFOWG), and hybrid weighted geometric (PFHWG) operators. Wei [6] and colleagues further developed generalized picture fuzzy aggregation operators based on the Hamacher operation, including PF Hamacher weighted aggregation, correlated weighted aggregation, induced correlated aggregation, prioritized aggregation, and power aggregation operators. Khan et al. [7] focused on examining logarithm PF weighted averaging, order weighted, hybrid weighted operators, as well as logarithm PF weighted geometric, order weighted, and hybrid weighted geometric operators. PF Dombi weighted averaging (PFDWA), PF Dombi order weighted (PFDOWA), PF Dombi hybrid Theweighted (PFDHWA), PF Dombi weighted geometric (PFDWG), and PF Dombi hybrid aggregation weighted geometric (PFDHWG) operators are some of the picture fuzzy operators combined with Dombi operation [8–13] that Jana et al. [14] proposed in their paper.

However, in many real-world problems, the issues can be too unclear or complex to be represented by people's intelligence and complex information. In some decision-making scenarios, precise or ambiguous figures are insufficient, and expressing the information in linguistic terms, such as "poor," "medium," or "good," is more appropriate. Zadeh [15] introduced the concept of linguistic variables, which was later followed by Herrera et al. [16], who discussed a consensus decision-making strategy using linguistic argumentation. Xu [17] developed a multi-attribute decision-making (MADM) approach using goal programming in linguistic information. Wang and Li [18] introduced the concept of intuitionistic linguistic fuzzy aggregation operators. Continuous linguistic terms were presented to researchers to prevent the loss of object information. Furthermore, Xu [19–22] introduced uncertain linguistic variables (ULVs) and provided some operational guidelines. Liu and Jin [23] proposed an application for multiple UL aggregation operators on IFS and introduced them. Meng et al. [24] introduced the IVIFUL Choquet averaging (IVIULCA) operator and IVIFUL Choquet geometric (IVIULCG) operator and used these operators to set up an

MADM problem with IVIULVs. Choquet aggregation operators using ULIVIFS arguments and operational, score, and accuracy functions for IVIULNs were also proposed. The concept of linguistic operator-based models has been explored in the context of picture fuzzy environments, as demonstrated by Qiyas et al. [25], who defined linguistic picture fuzzy sets (LPFS) operators and used them to create the MAGDM process. Liu and Zhang [26] introduced picture fuzzy linguistic numbers (PFLNs) and described the picture fuzzy linguistic weighted averaging (PFLWA) and weighted geometric (PFLWG) operators, which were used to model MAGDM problems. In the same setting, Qiyas et al. [27] developed MADM problems using linguistic picture fuzzy Dombi (LPFD) operators. In conclusion, interval-valued picture fuzzy uncertain linguistic variables (IVPFULVs) convey fuzzy information more accurately than LVs, and research on MADM problems with IVPFUL information is just beginning. Therefore, in this paper, we propose an IVPFULS and develop MADM problems where both the attribute weights and values take the form of IVPFULVs, based on the interval-valued picture fuzzy linguistic set proposed on the concept of uncertain linguistic set [28, 29]. First, we define the operating rules, score values, and correctness of IVPFULNs. The IVPFULWA operator and IVPFULWG operator are then developed. We also introduce the IVPFULDWA and IVPFULDWG operators and examine their desirable properties. Finally, a specific application of a numerical example is presented.

The remainder of this paper is structured as follows: Section 2 discusses fundamental PFN and ULV definitions and operations. Section 3 defines the IVPFUL set and provides certain IVPFULN operations. Interval-valued picture fuzzy uncertain linguistic weighted averaging (IVPFULWA) and interpolated picture fuzzy uncertain linguistic weighted geometric (IVPFULWG) operators are proposed in Section 4, along with some of their key properties. Section 5 introduces the interval-valued picture fuzzy uncertain linguistic Dombi average operator and the interval-valued picture fuzzy uncertain linguistic Dombigeometric operator. Two MADM approaches are constructed in Section 6 based on these two operators. In Section 7, a numerical example for evaluating mutual fund selection is provided. Finally, Section 8 offers concluding remarks.

2 Preliminaries

Here, it is important to quickly review some fundamental terms related to picture fuzzy sets (PFS), such interval-valued of picture fuzzy sets [3, 30].

2.1 Some Concept of Interval Picture Fuzzy Set

Definition1. [3, 4] Let PFS U be a fixed set X is written as

$$U = \{\langle \mu_U(x), \eta_U(x), \nu_U(x)\rangle | x \in X\},$$

where, positive be $\mu_U(x) \in [0,1]$, neutral be $\eta_U(x) \in [0,1]$ and negative be $\nu_U(x) \in [0,1]$ are membership degree, in a fuzzy set U where $0 \leq \mu_U(x) + \eta_U(x) + \nu_U(x) \leq 1$ for $x \in X$. Also, for refusal degree is for x as $\pi_U(x) = 1 - \mu_U(x) - \eta_U(x) - \nu_U(x)$. The pair (μ_U, η_U, ν_U) is named as picture fuzzy numbers (PFNs) or picture fuzzy values (PFVs).

2.2 Some Idea of Uncertain Linguistic Variables

This section addressed several concepts and operational rules that use LVs to introduce both qualitative and linguistic features [15, 16, 20, 21, 28, 29, 31]. Let $S = \{s_t | t = 1, 2, \ldots, p\}$ be a LTS with odd cardinality. Any stage, s_t represents a value for a linguistic variable and demonstrates the qualities listed below:
(i) Order set if $s_i \geq s_j$ if $i \geq j$
(ii) Negation operator if $neg(s_i) = s_j$ such that $j = t - i$
(iii) Max operator $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$
(iv) Min operator $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$. For example, the study [22] can be provided as:
$S = \{s_0 = extremely\ poor, s_1 = very\ poor, s_2 = poor, s_3 = medium, s_4 = good, s_5 = very\ good, s_6 = extremely\ good\}$.

We expanded the discrete term set S to a continuous term set to prevent information loss $S = \{s_t | s_0 \leq s_t \leq s_p, t \in [1, p]\}$, where p is an adequate size positive integer. If $s_t \in S$, it is referred to as a virtual LT, or an original linguistic term (LT). Decision-makers typically employ original LTS and virtual LTS solely used for computation to find alternatives and qualities [19–22, 32].

The input LTS may not fit any of the original linguistic labels and may instead be placed between any two of them, as is frequently observed in many real-world scenarios. In such situation, Xu [19–22] uncertain linguistic variables (ULT) were introduced, and some of their operational principles were supplied.

Definition2. [22] Let $s = [s_l, s_m]$, where $s_l, s_m \in S$, and s_l, s_m are the LVs s's lower and upper bounds, respectively. Also, let $\tilde{S}$ be the set of all ULTs. Let $s = [s_l, s_m]$, $s_1 = [s_{l_1}, s_{m_1}]$ and $s_2 = [s_{l_2}, s_{m_2}]$ be three ULVs, where $s, s_1, s_2 \in \tilde{S}$ and $\lambda \in [0,1]$, then operational laws of them defined as follows:
(i) $s_1 \oplus s_2 = [s_{l_1}, s_{m_1}] \oplus [s_{l_2}, s_{m_2}] = [s_{l_1} \oplus s_{l_2}, s_{m_1} \oplus s_{m_2}] = [s_{l_1+l_2}, s_{m_1+m_2}]$

(ii) $s_1 \otimes s_2 = [s_{l_1}, s_{m_1}] \otimes [s_{l_2}, s_{m_2}] = [s_{l_1} \otimes s_{l_2}, s_{m_1} \otimes s_{m_2}] = [s_{l_1 l_2}, s_{m_1 m_2}]$
(iii) $\lambda s = \lambda[s_l, s_m] = [\lambda s_l, \lambda s_m] = [s_{\lambda l}, s_{\lambda m}]$
(iv) $(s)^\lambda = ([s_l, s_m])^\lambda = [(s_l)^\lambda, (s_m)^\lambda] = [s_{l^\lambda}, s_{m^\lambda}]$.

3 Interval-Valued Picture Fuzzy Uncertain Linguistic Set (IVPFULS)

We introduce the IVLS and ULS to define INULS and IVPFULN based on the notions of INS, ULS, and INLS. This section includes the IVPFULN's operational guidelines and ranking order.

Definition3. Let Z be a fixed set and z represent the collective element within Z. The definition of IVPFULS p in Z is

$$p = \left\{ \left\langle z, s_{\phi(z)}, \mu_p(z), \eta_p(z), \nu_p(z) \right\rangle | z \in Z \right\} \tag{1}$$

where, $s_{\phi(z)} = [s_{\sigma(z)}, s_{\theta(z)}] \in S$, $\mu_p(z) = [\mu_p^l(z), \mu_p^u(z)] \subseteq [0,1]$, $\eta_p(z) = [\eta_p^l(z), \eta_p^u(z)] \subseteq [0,1]$, and $\nu_p(z) = [\nu_p^l(z), \nu_p^u(z)] \subseteq [0,1]$ with the condition $0 \leq \mu_p^u(z) + \eta_p^u(z) + \nu_p^u(z) \leq 1$. The functions $\mu_p(z)$, $\eta_p(z)$ and $\nu_p(z)$ are measured support, neutral, and objection membership values in an interval of an element z to the set Z to the ULVs $s_{\phi(z)} = [s_{\zeta(z)}, s_{\theta(z)}]$. For convenience, $p = \left\langle z, [s_{\zeta(p)}, s_{\theta(p)}], [\mu^l(p), \mu^u(p)], [\eta^l(p), \eta^u(p)], [\nu^l(p), \nu^u(p)] \right\rangle$ is the eight tuples called an IVPFULNs.

We defined some new operations on IVPFULNs:

Definition4. Let $p = \left\langle [s_{\sigma(p)}, s_{\theta(p)}], [\mu^l(p), \mu^u(p)], [\eta^l(p), \eta^u(p)], [\nu^l(p), \nu^u(p)] \right\rangle$ and $q = \left\langle [s_{\zeta(q)}, s_{\theta(q)}], [\mu^l(q), \mu^u(q)], [\eta^l(q), \eta^u(q)], [\nu^l(q), \nu^u(q)] \right\rangle$ be any two IVPFULNs, some operations of p and q defined for any real number $\lambda \in [0,1]$

(1) $p \oplus q = \Big\langle [s_{\zeta(p)+\zeta(q)}, s_{\theta(p)+\theta(q)}], [\mu^l(p) + \mu^l(q) - \mu^l(p)\mu^l(q), \mu^u(p) + \mu^u(q) - \mu^u(p)\mu^u(q)],$
$[\eta^l(p)\eta^l(q), \eta^u(p)\eta^u(q)], [\nu^l(p)\nu^l(q), \nu^u(p)\nu^u(q)] \Big\rangle$

(2) $p \otimes q = \Big\langle [s_{\zeta(p)\times\eta(q)}, s_{\theta(p)\times\theta(q)}], [\mu^l(p)\mu^l(q), \mu^u(p)\mu^u(q))],$
$[\eta^l(p) + \eta^l(q) - \eta^l(p)\eta^l(q), \eta^u(p) + \eta^u(q) - \eta^u(p)\eta^u(q)], [\nu^l(p) + \nu^l(q) - \nu^l(p)\nu^l(q), \nu^u(p) + \nu^u(q) -$
$\nu^u(p)\nu^u(q)] \Big\rangle$

(3) $\lambda p = \Big\langle [s_{\lambda\zeta(p)}, s_{\lambda\theta(p)}], [1 - (1 - \mu^l(p))^\lambda, 1 - (1 - \mu^u(p))^\lambda], [{\eta^l}^\lambda(p), {\eta^u}^\lambda(p)], [{\nu^l}^\lambda(p), {\nu^u}^\lambda(p)] \rangle$

(4) $p^\lambda = \Big\langle [s_{\zeta(p)^\lambda}, s_{\theta(p)^\lambda}], {\mu^u}^\lambda(p)], [1 - (1 - \eta^l(p))^\lambda, 1 - (1 - \eta^u(p))^\lambda], [1 - (1 - \nu^l(p))^\lambda, 1 - (1 - \nu^u(p))^\lambda] \rangle$.

Definition5. Let p and q be any two IVPFULNs, then

(1) $p + q = q + p$

(2) $p.q = q.p$

(3) $\lambda(p + q) = \lambda p + \lambda q$, for $\lambda \in [0,1]$

(4) $(p.q)^\lambda = p^\lambda + q^\lambda$, for $\lambda \in [0,1]$

(5) $\lambda_1 p + \lambda_2 p = (\lambda_1 + \lambda_2)p$, for $\lambda_1, \lambda_2 \in [0,1]$

(6) $p^{\lambda_1}.p^{\lambda_2} = p^{\lambda_1+\lambda_2}$, for $\lambda_1, \lambda_2 \in [0,1]$

(7) $(p + q) + r = p + (q + r)$

(8) $(p.q).r = p.(q.r)$.

Based on the definition of score and accuracy function in the study [24] defined on interval-valued intuitionistic uncertain linguistic (IVIULNs) numbers, we defined score and accuracy on an interval neutrosophic uncertain linguistic information defined below.

Definition6. Let $p = \left\langle [s_{\sigma(p)}, s_{\theta(p)}], [\mu^l(p), \mu^u(p)], [\eta^l(p), \eta^u(p)], [\nu^l(p), \nu^u(p)] \right\rangle$ be any IVPFULN. Then, defined score function of p is $\Lambda(p)$ by

$$\Lambda(p) = s_{\frac{(\sigma(p)+\theta(p))(2+\mu^l(p)+\mu^u(p)-\eta^l(p)-\eta^u(p)-\nu^l(p)-\nu^u(p))}{4}}, \ \Lambda(p) \in [0,1] \tag{2}$$

The accuracy function of p is $\Phi(p)$ by

$$\Phi(p) = s_{\frac{(\sigma(p)+\theta(p))(\eta^l(p)+\eta^u(p)+\nu^l(p)+\nu^u(p))}{4}}, \ \Phi(p) \in [0,1] \tag{3}$$

The following is a definition of prioritised analysis between any two IVPFULNs p and q based on the aforementioned design of score and accuracy:

(i) If $\Lambda(p) < \Lambda(q)$, imply $p \prec q$
(ii) If $\Lambda(p) > \Lambda(q)$, imply $p \succ q$
(iii) If $\Lambda(p) = \Lambda(q)$, then
  (1) If $\Phi(p) < \Phi(q)$, imply $p \prec q$.
  (2) If $\Phi(p) > \Phi(q)$, imply $p \succ q$.
  (3) If $\Phi(p) = \Phi(q)$, imply $p \sim q$.

4 Interval-Valued Picture Fuzzy Uncertain Linguistic Aggregation Operators

Here we defined IVPFULWA operator and study some of its properties.

4.1 IVPFULWA Operator

Definition7. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$. Then interval-valued picture fuzzy uncertain linguistic weighted average (IVPFULWA) function $IVPFULWA : \times^{\zeta} \to \times$ defined as follows:

$$IVPFULWA_{\varpi}(p_1, p_2, \ldots, p_{\zeta}) = \bigoplus_{b=1}^{\zeta} (\psi_b p_b) \tag{4}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $p_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

By the operations on IVPFULNs , we derive the following theorem.

Theorem1. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$, then aggregating values of IVPFULNs p_b $(b = 1, 2, \ldots, \zeta)$ is also an IVPFULN, and further,

$$\begin{aligned} IVPFULWA_{\psi}(p_1, p_2, \ldots, p_{\zeta}) = \bigoplus_{b=1}^{\zeta} (\psi_b p_b) = \Bigg\langle \Bigg[s_{\sum\limits_{b=1}^{\zeta} \psi_b \eta(p_b)}, s_{\sum\limits_{b=1}^{\zeta} \psi_b \theta(p_b)} \Bigg], \\ \Bigg[1 - \prod_{b=1}^{\zeta} (1 - \mu^l(p_b))^{\psi_b}, 1 - \prod_{b=1}^{\zeta} (1 - \mu^u(p_b))^{\psi_b} \Bigg], \Bigg[\prod_{b=1}^{\zeta} (\eta^l(p_b))^{\psi_b}, \prod_{b=1}^{\zeta} (\eta^l(p_b))^{\psi_b} \Bigg], \\ \Bigg[\prod_{b=1}^{\zeta} (\nu^l(p_b))^{\psi_b}, \prod_{b=1}^{\zeta} (\nu^l(\varrho_b))^{\varpi_b} \Bigg] \Bigg\rangle \end{aligned} \tag{5}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

Proof:

We prove the Eq. (7) below using mathematical induction.

(i) When $\zeta = 2$, we get

$$\Bigg\langle \Bigg[s_{\psi_b \sigma(p_b)}, s_{\psi_b \theta(p_b)} \Bigg], \Big[1 - (1 - \mu^l(p_b))^{\psi_b}, 1 - (1 - \mu^u(p_b))^{\psi_b} \Big], \Big[(\eta^l(p_b))^{\psi_b}, (\eta^l(p_b))^{\psi_b} \Big], \Big[(\nu^l(p_b))^{\psi_b}, (\nu^l(p_b))^{\psi_b} \Big] \Bigg\rangle$$

for $b = 1, 2$.

Then,

$$\begin{aligned} IVPFULWA_{\psi}(p_1,p_2) = \bigoplus_{b=1}^{2}\psi_b p_b = \left\langle \left[s_{\sum\limits_{b=1}^{2}\psi_b \eta(p_b)}, s_{\sum\limits_{b=1}^{2}\psi_b \theta(p_b)} \right], \right. \\ \left[1-\prod_{b=1}^{2}(1-\mu^l(p_b))^{\psi_b}, 1-\prod_{b=1}^{2}(1-\mu^u(p_b))^{\psi_b}\right], \left[\prod_{b=1}^{2}(\eta^l(p_b))^{\psi_b}, \prod_{b=1}^{2}(\eta^l(p_b))^{\psi_b}\right], \\ \left.\left[\prod_{b=1}^{2}(\nu^l(p_b))^{\psi_b}, \prod_{b=1}^{2}(\nu^l(p_b))^{\psi_b}\right]\right\rangle \end{aligned} \tag{6}$$

(ii) Hypothesis, Eq. (7) holds for $\zeta = k$ $(k \geq 2)$, then

$$\begin{aligned} IVPFULWA_{\psi}(p_1,p_2,\ldots,p_k) = \bigoplus_{b=1}^{k}(\psi_b p_b) = \left\langle \left[s_{\sum\limits_{b=1}^{k}\psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k}\psi_b \theta(p_b)} \right], \right. \\ \left[1-\prod_{b=1}^{k}(1-\mu^l(p_b))^{\psi_b}, 1-\prod_{b=1}^{k}(1-\mu^u(p_b))^{\psi_b}\right], \left[\prod_{b=1}^{k}(\eta^l(p_b))^{\psi_b}, \prod_{b=1}^{k}(\eta^l(p_b))^{\psi_b}\right], \\ \left.\left[\prod_{b=1}^{k}(\nu^l(p_b))^{\psi_b}, \prod_{b=1}^{k}(\nu^l(p_b))^{\psi_b}\right]\right\rangle \end{aligned} \tag{7}$$

When $b = k+1$, we get

$$\begin{aligned} IVPFULWA_{\psi}(p_1,p_2,\ldots,p_{k+1},p_k) = \bigoplus_{b=1}^{k}(\psi_b p_b) = \left\langle \left[s_{\sum\limits_{b=1}^{k}\psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k}\psi_b \theta(p_b)} \right], \right. \\ \left[1-\prod_{b=1}^{k}(1-\mu^l(p_b))^{\psi_b}, 1-\prod_{b=1}^{k}(1-\mu^u(p_b))^{\psi_b}\right], \left[\prod_{b=1}^{k}(\eta^l(p_b))^{\psi_b}, \prod_{b=1}^{k}(\eta^l(p_b))^{\psi_b}\right], \\ \left.\left[\prod_{b=1}^{k}(\nu^l(p_b))^{\psi_b}, \prod_{b=1}^{k}(\nu^l(p_b))^{\psi_b}\right]\right\rangle \\ \bigoplus \left\langle \left[s_{\psi_{k+1}\sigma(p_{k+1})}, s_{\psi_{k+1}\theta(p_{k+1})} \right], \left[1-(1-\mu^l(p_{k+1}))^{\psi_{k+1}}, 1-(1-\mu^u(p_{k+1}))^{\psi_{k+1}}\right], \right. \\ \left. \left[(\eta^l(p_{k+1}))^{\psi_{k+1}}, (\eta^u(p_{k+1}))^{\psi_{k+1}}\right], \left[(\nu^l(p_{k+1}))^{\psi_{k+1}}, (\nu^u(p_{k+1}))^{\psi_{k+1}}\right] \right\rangle = \left\langle \left[s_{\sum\limits_{b=1}^{k+1}\psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k+1}\psi_b \theta(p_b)} \right], \right. \\ \left[1-\prod_{b=1}^{k+1}(1-\mu^l(p_b))^{\psi_b}, 1-\prod_{b=1}^{k+1}(1-\mu^u(p_b))^{\psi_b}\right], \left[\prod_{b=1}^{k+1}(\eta^l(p_b))^{\psi_b}, \prod_{b=1}^{k+1}(\eta^u(p_b))^{\psi_b}\right], \\ \left.\left[\prod_{b=1}^{k+1}(\nu^l(p_b))^{\psi_b}, \prod_{b=1}^{k+1}(\nu^u(p_b))^{\psi_b}\right]\right\rangle \end{aligned} \tag{8}$$

Thus, for $\zeta = k+1$, Eq. (7) holds, and results is obtained.

Theorem2. (Idempotent Property)
Let $p_b = \left\langle [s_{\eta(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), T^u(p_b)], [I^l(p_b), I^u(p_b)], [F^l(p_b), F^u(p_b)] \right\rangle$ be a set of INULNs for $(b = 1,2,\ldots,\zeta)$ are equal, i.e., $p_b = p$ for all b. Then

$$INULWA_{\psi}(p_1,p_2,\ldots,p_{\zeta}) = p \tag{9}$$

Theorem3. (Boundedness Property)
Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1,2,\ldots,\zeta)$.
Let $s_{\sigma}^{-} = \min\limits_{1\leq b\leq \zeta}\{s_{\sigma(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$ $s_{\sigma}^{+} = \max\limits_{1\leq b\leq \zeta}\{s_{\sigma(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$,

$s_{\theta}^{-} = \min\limits_{1 \leq b \leq \zeta} \{s_{\theta(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$ and $s_{\theta}^{+} = \max\limits_{1 \leq b \leq \zeta} \{s_{\theta(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$.
Let $\mu^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\mu_b^l | [\mu_b^l, \mu_b^u] \in p_b\}$, and $\mu^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\mu^u(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$
and $\mu^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\mu^l(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$, and $\mu^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\mu^u(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$.
Let $\eta^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\eta^l(p_b) | [\eta^l(p_b), \eta^u(p_b)] \in p_b\}$, and $\eta^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\eta^u(p_b) | [\eta_b^l, \eta^u(p_b)] \in p_b\}$
and $\eta^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\eta^l(p_b) | [\eta^l(p_b), \eta^u(p_b)] \in p_b\}$, and $\eta^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\eta^u(p_b) | [\eta_b^l(p_b), \eta^u(p_b)] \in p_b\}$.
Let $\nu^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\nu_b^l | [\nu_b^l(p_b), \nu^u(p_b)] \in p_b\}$, and $\nu^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\nu^u(p_b) | [\nu_b^l(p_b), \nu^u(p_b)] \in p_b\}$
and $\nu^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\nu^l(p_b) | [\nu^l(p_b), \nu^u(p_b)] \in p_b\}$, and $\nu^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\nu^u(p_b) | [\nu_b^l(p_b), \nu^u(p_b)] \in p_b\}$,
for all b, then we have

$$\begin{aligned} \{[s_{\sigma}^{-}, s_{\theta}^{-}], [\mu^{l-}, \mu^{u-}], [\eta^{l-}, \eta^{u-}], [\nu^{l-}, \nu^{u-}]\} &\leq IVPFULWA_{\psi}(p_1, p_2, \ldots, p_{\zeta}) \\ &\leq \{[s_{\sigma}^{+}, s_{\theta}^{+}], [\mu^{l+}, \mu^{u+}], [\eta^{l+}, \eta^{u+}], [\nu^{l+}, \nu^{u+}]\}. \end{aligned}$$

Theorem4. (Monotonicity Property)
Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ and
$p_b' \left\langle [s'_{\sigma(p_b')}, s'_{\theta(p_b')}], [\mu^{'l}(p_b'), \mu^{'u}(p_b')], [\eta^{'l}(p_b'), \eta^{'u}(p_b')], [\nu^{'l}(p_b'), \nu^{'u}(p_b')] \right\rangle$ be two sets of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$.
If $p_b \leq p_b'$ for all b, then

$$IVPFULWA_{\psi}(p_1, p_2 \ldots, p_{\zeta}) \leq IVPFULWA_{\psi}(p_1^{'}, p_2^{'}, \ldots, p_{\zeta}^{'}) \tag{10}$$

4.2 IVPFULWG Operator

Now, we will introduce interval-valued picture fuzzy uncertain linguistic weighted geometric (IVPFULWG) operator and its properties.

Definition8. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$. Then interval-valued picture fuzzy uncertain linguistic weighted geometric (IVPFULWG) function $IVPFULWG : \times^{\zeta} \rightarrow \times$ defined as follows:

$$IVPFULWG_{\psi}(p_1, p_2, \ldots, p_{\zeta}) = \bigotimes_{b=1}^{\zeta} (p_b)^{\psi_b} \tag{11}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

By the operations on IVPFULNs , we derive the following theorem.

Theorem5. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $b = 1, 2, \ldots, \zeta$, then aggregating values of IVPFULNs p_b for $b = 1, 2, \ldots, \zeta$ using IVPFULWG operator is also an IVPFULN, and further,

$$\begin{aligned} IVPFULWG_{\psi}(p_1, p_2, \ldots, p_{\zeta}) = \bigotimes_{b=1}^{\zeta} (p_b)^{\psi_b} = \Bigg\langle \Bigg[s_{\prod\limits_{b=1}^{\zeta} (\sigma(p_b))^{\psi_b}}, s_{\prod\limits_{b=1}^{\zeta} (\theta(p_b))^{\psi_b}} \Bigg], \\ \Big[\prod_{b=1}^{\zeta} (\mu^l(p_b))^{\psi_b}, \prod_{b=1}^{\zeta} (\mu^u(p_b))^{\psi_b} \Big], \Big[1 - \prod_{b=1}^{\zeta} (1 - \eta^l(p_b))^{\psi_b}, 1 - \prod_{b=1}^{\zeta} (1 - \eta^u(p_b))^{\psi_b} \Big], \\ \Big[1 - \prod_{b=1}^{\zeta} (1 - \nu^l(p_b))^{\psi_b}, 1 - \prod_{b=1}^{\zeta} (1 - F^u(p_b))^{\psi_b} \Big] \Bigg\rangle \end{aligned} \tag{12}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

Theorem6. (Idempotent Property)
Let $p_b = \left\langle [s_{\eta(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), T^u(p_b)], [I^l(p_b), I^u(p_b)], [F^l(p_b), F^u(p_b)] \right\rangle$ be a set of INULNs for $(b = 1, 2, \ldots, \zeta)$ are equal, i.e., $p_b = p$ for all b. Then

$$INULWG_{\psi}(p_1, p_2, \ldots, p_{\zeta}) = p \tag{13}$$

Theorem7. (Boundedness Property)
Let $p_b = \Big\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \Big\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$.
Let $s_\sigma^- = \min\limits_{1 \leq b \leq \zeta} \{s_{\sigma(p_b)} | [s_{\eta(p_b)}, s_{\theta(p_b)}] \in p_b\}$
$s_\sigma^+ = \max\limits_{1 \leq b \leq \zeta} \{s_{\sigma(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$,
$s_\theta^- = \min\limits_{1 \leq b \leq \zeta} \{s_{\theta(p_b)} | [s_{\eta(p_b)}, s_{\theta(p_b)}] \in p_b\}$ and $s_\theta^+ = \max\limits_{1 \leq b \leq \zeta} \{s_{\theta(p_b)} | [s_{\sigma(p_b)}, s_{\theta(p_b)}] \in p_b\}$.
Let $\mu^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\mu_b^l | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$,
and $\mu^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\mu^u(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$ and $\mu^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\mu^l(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$,
and $\mu^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\mu^u(p_b) | [\mu^l(p_b), \mu^u(p_b)] \in p_b\}$.
Let $\eta^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\eta^l(p_b) | [\eta^l(p_b), \eta^u(p_b)] \in p_b\}$,
and $\eta^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\eta^u(p_b) | [\eta_b^l(p_b), I^u(p_b)] \in p_b\}$ and $\eta^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\eta^l(p_b) | [\eta^l(p_b), \eta^u(p_b)] \in p_b\}$,
and $\eta^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\eta^u(p_b) | [\eta^l(p_b), \eta^u(p_b)] \in p_b\}$.
Let $\nu^{l-} = \min\limits_{1 \leq b \leq \zeta} \{\nu_b^l | [\nu^l(p_b), \nu^u(p_b)] \in p_b\}$, and
$\nu^{u-} = \min\limits_{1 \leq b \leq \zeta} \{\nu^u(p_b) | [\nu^l(p_b), \nu^u(p_b)] \in p_b\}$ and $\nu^{l+} = \max\limits_{1 \leq b \leq \zeta} \{\nu^l(p_b) | [\nu^l(p_b), \nu^u(p_b)] \in p_b\}$,
and $\nu^{u+} = \max\limits_{1 \leq b \leq \zeta} \{\nu^u(p_b) | [\nu^l(p_b), \nu^u(p_b)] \in p_b\}$, for all b, then we have

$$\begin{aligned}\{[s_\sigma^-, s_\theta^-], [\mu^{l-}, \mu^{u-}], [\eta^{l-}, \eta^{u-}], [\nu^{l-}, \nu^{u-}]\} &\leq IVPFULWG_\psi(p_1, p_2, \ldots, p_\zeta) \\ &\leq \{[s_\sigma^+, s_\theta^+], [\mu^{l+}, \mu^{u+}], [\eta^{l+}, \eta^{u+}], [\nu^{l+}, \nu^{u+}]\}.\end{aligned}$$

Theorem8. (Monotonicity Property)
Let $p_b = \Big\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \Big\rangle$
and $p'_b \Big\langle [s'_{\sigma(p'_b)}, s'_{\theta(p'_b)}], [\mu^{'l}(p'_b), \mu^{'u}(p'_b)], [\eta^{'l}(p'_b), \eta^{'u}(p'_b)], [\nu^{'l}(p'_b), \nu^{'u}(p'_b)] \Big\rangle$ be two sets of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$. If $p_b \leq p'_b$ for all b, then

$$IVPFULWG_\psi(p_1, p_2 \ldots, p_\zeta) \leq IVPFULWG_\psi(p_1^{'}, p_2^{'}, \ldots, p_\zeta^{'}) \tag{14}$$

5 IVPFUL Dombi Aggregation Operators

5.1 IVPFULDWA Operator

Definition9. Let $p_b = \Big\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \Big\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$. The IVPFULDWA function $IVPFULDWA : \times^\zeta \to \times$ defined as follows:

$$IVPFULDWA_\psi(p_1, p_2, \ldots, p_\zeta) = \bigoplus_{b=1}^{\zeta} (\psi_b p_b) \tag{15}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_\zeta)^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

By the operations on IVPFULNs , we derive the following theorem.

Theorem9. Let $p_b = \Big\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \Big\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$, then aggregating values using IVPFULDWA opearator p_b $(b = 1, 2, \ldots, \zeta)$ is also an

IVPFULN, and further,

$$\begin{aligned} IVPFULDWA_{\psi}(p_1,p_2,\ldots,p_{\zeta}) = \bigoplus_{b=1}^{\zeta}(\psi_b p_b) = \Bigg\langle \Bigg[s_{\sum\limits_{b=1}^{\zeta}\psi_b\sigma(p_b)}, s_{\sum\limits_{b=1}^{\zeta}\psi_b\theta(p_b)} \Bigg], \\ \Bigg[1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\mu^l(p_b)}{1-\mu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, 1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\mu^u(p_b)}{1-\mu^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg], \\ \Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\eta^l(p_b)}{\eta^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, \frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\eta^u(p_b)}{\eta^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg], \\ \Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\nu^l(p_b)}{\nu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, \frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\nu^u(p_b)}{\nu^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg]\Bigg\rangle \end{aligned} \tag{16}$$

where, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \ldots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

Proof:

We prove the Eq. (18) below using mathematical induction.

(i) When $b = 2$, we get

$$\Bigg\langle \Big[s_{\psi_b\eta(p_b)}, s_{\psi_b\theta(p_b)} \Big], \Bigg[1-\frac{1}{1+\Big\{\psi_b\Big(\frac{\mu^l(p_b)}{1-\mu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, 1-\frac{1}{1+\Big\{\psi_b\Big(\frac{\mu^u(p_b)}{1-\mu^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg],$$

$$\Bigg[\frac{1}{1+\Big\{\psi_b\Big(\frac{1-\eta^l(p_b)}{\eta^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, \frac{1}{1+\Big\{\psi_b\Big(\frac{1-\eta^u(p_b)}{\eta^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg], \Bigg[\frac{1}{1+\Big\{\psi_b\Big(\frac{1-\nu^l(p_b)}{\nu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, 1-\frac{1}{1+\Big\{\psi_b\Big(\frac{1-\nu^u(p_b)}{\nu^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg]\Bigg\rangle$$

for $\xi = 1, 2$.

Then,

$$\begin{aligned} IVPFULDWA_{\psi}(p_1,p_2) = \bigoplus_{b=1}^{2}\psi_b p_b = \Bigg\langle \Bigg[s_{\sum\limits_{b=1}^{2}\psi_b\varrho(p_b)}, s_{\sum\limits_{b=1}^{2}\psi_b\theta(p_b)} \Bigg], \\ \Bigg[1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{\mu^l(p_b)}{1-\mu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, 1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{T^u(p_b)}{1-T^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg], \\ \Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{1-\eta^l(p_b)}{\eta^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, \frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{1-\eta^u(p_b)}{\eta^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg], \\ \Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{1-\nu^l(p_b)}{\nu^l(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}, \frac{1}{1+\Big\{\sum\limits_{b=1}^{2}\psi_b\Big(\frac{1-\nu^u(p_b)}{\nu^u(p_b)}\Big)^{\varrho}\Big\}^{1/\varrho}}\Bigg]\Bigg\rangle. \end{aligned} \tag{17}$$

(ii) Hypothesis, Eq. (18) holds for $\zeta = k$ $(k \geq 2)$, then

$$\begin{aligned} IVPFULDWA_{\psi}(p_1, p_2, \ldots, p_k) = \bigoplus_{b=1}^{k}(\psi_b p_b) = \Bigg\langle \left[s_{\sum\limits_{b=1}^{k} \psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k} \psi_b \theta(p_b)} \right], \\ \left[1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{\mu^l(p_b)}{1-\mu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{\mu^u(p_b)}{1-\mu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\eta^l(p_b)}{\eta^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\eta^u(p_b)}{\eta^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\nu^l(p_b)}{\nu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\nu^u(p_b)}{\nu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right] \Bigg\rangle. \end{aligned} \tag{18}$$

When $\tau = k + 1$, we get

$$IVPFULDWA_{\psi}(p_1, p_2, \ldots, p_k, p_{k+}) = \bigoplus_{b=1}^{k}(\psi_b p_b) \bigoplus (\psi_{k+1} p_{k+1})$$

$$\begin{aligned} = \Bigg\langle \left[s_{\sum\limits_{b=1}^{k} \psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k} \psi_b \theta(p_b)} \right], \left[1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{\mu^l(p_b)}{1-\mu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{\mu^u(p_b)}{1-\mu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\eta^l(p_b)}{\eta^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\eta^u(p_b)}{\eta^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\nu^l(p_b)}{\nu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k} \psi_b \left(\frac{1-\nu^u(p_b)}{\nu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right] \Bigg\rangle \end{aligned}$$

$$\begin{aligned} \oplus \Bigg\langle \left[s_{\psi_{k+1}\sigma(p_{k+1})}, s_{\psi_{k+1}\theta(p_{k+1})} \right], \left[1 - \frac{1}{1+\left\{ \psi_{k+1} \left(\frac{\mu^l(p_{k+1})}{1-\mu^l(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1+\left\{ \psi_{k+1} \left(\frac{\mu^u(p_{k+1})}{1-\mu^u(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1+\left\{ \psi_{k+1} \left(\frac{1-\eta^l(p_{k+1})}{I^l(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1+\left\{ \psi_{k+1} \left(\frac{1-\eta^u(p_{k+1})}{\eta^u(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}} \right], \left[\frac{1}{1+\left\{ \psi_{k+1} \left(\frac{1-\nu^l(p_{k+1})}{\nu^l(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}}, 1- \frac{1}{1+\left\{ \psi_{k+1} \left(\frac{1-\nu^u(p_{k+1})}{\nu^u(p_{k+1})} \right)^{\varrho} \right\}^{1/\varrho}} \right] \Bigg\rangle \end{aligned}$$

$$\begin{aligned} = \Bigg\langle \left[s_{\sum\limits_{b=1}^{k+1} \psi_b \sigma(p_b)}, s_{\sum\limits_{b=1}^{k+1} \psi_b \theta(p_b)} \right], \left[1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{\mu^l(p_b)}{1-\mu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{\mu^u(p_b)}{1-\mu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{1-\eta^l(p_b)}{\eta^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{1-\eta^u(p_b)}{\eta^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right], \\ \left[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{1-\nu^l(p_b)}{\nu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{k+1} \psi_b \left(\frac{1-\nu^u(p_b)}{\nu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \right] \Bigg\rangle. \end{aligned} \tag{19}$$

Thus, for $\zeta = k + 1$, Eq. (18) holds, and results is obtained.

5.2 IVPFULDWG Operator

Definition10. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1, 2, \ldots, \zeta)$. Then interval-valued picture fuzzy uncertain linguistic Dombi weighted average

(IVPFULDWG) function $INULDWG : \times^{\zeta} \rightarrow \times$ defined as follows:

$$IVPFULDWG_{\psi}(p_1, p_2, \dots, p_{\zeta}) = \bigotimes_{b=1}^{\zeta} (p_b)^{\psi_b} \tag{20}$$

where, $\psi = (\psi_1, \psi_2, \dots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \dots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

In view of Dombi operation on IVPFULNs , we derive the following theorem.

Theorem10. Let $p_b = \left\langle [s_{\sigma(p_b)}, s_{\theta(p_b)}], [\mu^l(p_b), \mu^u(p_b)], [\eta^l(p_b), \eta^u(p_b)], [\nu^l(p_b), \nu^u(p_b)] \right\rangle$ be a set of IVPFULNs for $(b = 1, 2, \dots, \zeta)$, then aggregating values of IVPFULNs p_b $(b = 1, 2, \dots, \zeta)$ is also an IVPFULN, and further,

$$\begin{aligned} IVPFULDWG_{\psi}(p_1, p_2, \dots, p_{\zeta}) = \bigotimes_{b=1}^{\zeta} (\psi_b p_b) = \Bigg\langle \Bigg[s_{\prod\limits_{b=1}^{\zeta} (\eta(p_b))^{\psi_b}}, s_{\prod\limits_{b=1}^{\zeta} (\theta(p_b))^{\psi_b}} \Bigg], \\ \Bigg[\frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{1-\mu^l(p_b)}{\mu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, \frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{1-\mu^u(p_b)}{\mu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \Bigg], \\ \Bigg[1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{\eta^l(p_b)}{1-\eta^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{\eta^u(p_b)}{1-\eta^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \Bigg], \\ \Bigg[1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{F^l(p_b)}{1-\nu^l(p_b)} \right)^{\varrho} \right\}^{1/\varrho}}, 1 - \frac{1}{1 + \left\{ \sum\limits_{b=1}^{\zeta} \psi_b \left(\frac{F^u(p_b)}{1-\nu^u(p_b)} \right)^{\varrho} \right\}^{1/\varrho}} \Bigg] \Bigg\rangle \end{aligned} \tag{21}$$

where, $\psi = (\psi_1, \psi_2, \dots, \psi_{\zeta})^T$ be followed the weight vector of p_b $(b = 1, 2, \dots, \zeta)$, with $\psi_b \in [0, 1]$, and $\sum\limits_{b=1}^{\zeta} \psi_b = 1$.

Proof: This theorem can be proved easily. ☐

6 Model for MADM Method With INUL Information

The weights of the characteristics are real values under IPUL information in the MADM technique that we propose in this work, which uses INUL aggregation operators. Here, the MADM technique is utilised to assess the utility of choosing an index of rural development under ambiguous language interval data. Let $Q = \{Q_1, Q_2, \dots, Q_{\zeta}\}$ be a finite set of alternatives, and $G = \{G_1, G_2, \dots, G_{\zeta}\}$ be a set of attributes. Let $\psi = (\psi_1, \psi_2, \dots, \psi_{\zeta})^T$ be the weight vector for the attribute b_j $(b = 1, 2, \dots, \zeta)$ that are known such that $\psi_b \in [0, 1]$, where $\sum\limits_{b=1}^{\zeta} \psi_b = 1$. Suppose that $Q = (a_{\rho})_{\zeta \times \zeta}$ is the INUL decision matrix, where $p_{\rho} = ([s_{\sigma(p_{ab})}, s_{\theta(p_{ab})}], [\mu^l(p_{ab}), \mu^u(p_{\rho})], [\eta^l(p_{ab}), \eta^u(p_{ab})], [\nu^l(p_{ab}), \nu^u(p_{ab})])$ is the IVPFULN for the alternative $p_{ab} \in Q$ w.r.t. the attribute $p_b \in G$.

The approach uses the IVPFULWA and IVPFULWG operators to interpret the MADM issue with IVPFUL information.

Algorithm

Input: To the selection of desirable alternatives.

Output: Best alternative.

Case 1

Step 1. We make use of the decision-making data presented in matrix A and the IVPULWA operator.

$$IVPFULWA_{\psi}(p_{11},p_{12},\dots,p_{1\zeta})=\bigoplus_{b=1}^{\zeta}(\psi_b p_{ab})\Upsilon_a=\Bigg\langle\Bigg[s_{\sum\limits_{b=1}^{\zeta}\psi_b\eta(p_{ab})},s_{\sum\limits_{b=1}^{\zeta}\psi_b\theta(p_{ab})}\Bigg],$$
$$\Big[1-\prod_{b=1}^{\zeta}(1-\mu^{l}(p_{ab}))^{\psi_b},1-\prod_{b=1}^{\zeta}(1-\mu^{u}(p_{ab}))^{\psi_b}\Big],$$
$$\Big[\prod_{b=1}^{\zeta}(\eta^{l}(p_{ab}))^{\psi_b},\prod_{b=1}^{\zeta}(\eta^{l}(p_{ab}))^{\psi_b}\Big],\Big[\prod_{b=1}^{\zeta}(\nu^{l}(p_{ab}))^{\psi_b},\prod_{b=1}^{\zeta}(\nu^{l}(p_{ab}))^{\psi_b}\Big]\Bigg\rangle \tag{22}$$

or

$$IVPFULWG_{\psi}(p_{11},p_{12},\dots,p_{1\zeta})=\bigotimes_{b=1}^{\zeta}(p_{ab})^{\psi_b}\Upsilon_a=\Bigg\langle\Bigg[s_{\prod\limits_{b=1}^{\zeta}(\sigma(p_{\psi_b}))^{\psi_b}},s_{\prod\limits_{b=1}^{\zeta}(\theta(p_{ab}))^{\psi_b}}\Bigg],$$
$$\Big[\prod_{b=1}^{\zeta}(\mu^{l}(_{\psi_b}))^{\psi_b},\prod_{b=1}^{\zeta}(\mu^{u}(p_{ab}))^{\psi_b}\Big],\Big[1-\prod_{b=1}^{\zeta}(1-\eta^{l}(p_{ab}))^{\psi_b},1-\prod_{b=1}^{\zeta}(1-\eta^{u}(p_{ab}))^{\psi_b}\Big],$$
$$\Big[1-\prod_{b=1}^{\zeta}(1-\nu^{l}(p_{ab}))^{\psi_b},1-\prod_{b=1}^{\zeta}(1-\nu^{u}(p_{ab}))^{\psi_b}\Big]\Bigg\rangle \tag{23}$$

Case 2
If we applied IVPFULDWA (IVPFULDWG) operator, then get the scheme as follows:

$$IVPFULDWA_{\psi}(p_1,p_2,\dots,p_{\zeta})=\bigoplus_{b=1}^{\zeta}(\psi_b p_b)=\Bigg\langle\Bigg[s_{\sum\limits_{b=1}^{\zeta}\psi_b\sigma(p_b)},s_{\sum\limits_{b=1}^{\zeta}\psi_b\theta(p_b)}\Bigg],$$
$$\Bigg[1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\mu^{l}(p_b)}{1-\mu^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\mu^{u}(p_b)}{1-\mu^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg],$$
$$\Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\eta^{l}(p_b)}{\eta^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\eta^{u}(p_b)}{\eta^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg],$$
$$\Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\nu^{l}(p_b)}{\nu^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\nu^{u}(p_b)}{\nu^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg]\Bigg\rangle \tag{24}$$

or

$$IVPFULDWG_{\psi}(p_1,p_2,\dots,p_{\zeta})=\bigotimes_{b=1}^{\zeta}(p_b)^{\psi_b}=\Bigg\langle\Bigg[s_{\prod\limits_{b=1}^{\zeta}(\sigma(p_b))^{\psi_b}},s_{\prod\limits_{b=1}^{\zeta}(\theta(p_b))^{\psi_b}}\Bigg],$$
$$\Bigg[\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\mu^{l}(p_b)}{\mu^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{1-\nu^{u}(p_b)}{\nu^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg],$$
$$\Bigg[1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\eta^{l}(p_b)}{1-\eta^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\eta^{u}(p_b)}{1-\eta^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg],$$
$$\Bigg[1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\nu^{l}(p_b)}{1-\nu^{l}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}},1-\frac{1}{1+\Big\{\sum\limits_{b=1}^{\zeta}\psi_b\Big(\frac{\nu^{u}(p_b)}{1-\nu^{u}(p_b)}\Big)^{\sigma}\Big\}^{1/\sigma}}\Bigg]\Bigg\rangle \tag{25}$$

to obtained the overall values Υ_{ρ} $(\rho=1,2,\dots,\zeta)$ of the alternative p_b.

Step 2. Ranking all of the options p_b is done by evaluating the score $\Lambda(\Upsilon_a)$ $(a = 1, 2, \ldots, \xi)$ based on the total IVPFUL information.To get the preferred option Q_a, perform $(b = 1, 2, \ldots, \zeta)$. If the value of $\Lambda(\Upsilon_a)$ and $\Lambda(\Upsilon_b)$ are same, then we next proceed to evaluate degrees of accuracy $\Phi(\Upsilon_a)$ and $\Lambda(\Upsilon_b)$ rest on overall IVPFUL information of Υ_a and Υ_b, and rank the alternative Q_a depending with the accuracy $\Phi(\Upsilon_a)$ and $\Phi(\Upsilon_b)$.

Step 3. In order to select the best option(s) in accordance with $\Lambda(\Upsilon_a)$ $(a = 1, 2, \ldots, \xi)$, rank all of the alternatives Q_a.

Step 4. Stop.

7 Numerical Example

7.1 Application

The decision-making process has been illustrated in the following with a numerical example relating investment choice to the suitability of the suggested MADM challenges. A potential investor wants to put money into a mutual fund business. Before making an investment, a potential investor could investigate five mutual fund companies as possibilities, including:

(Q_1): Large cap fund
(Q_2): Liquid fund
(Q_3): Blue chip fund
(Q_4): Hybrid fund.

The expert team examined the mutual funds (alternatives) in light of the five characteristics listed below and provided recommendations.

(G_1): Short term
(G_2): Mid term
(G_3): Long term
(G_4): Risk of the funds
(G_5): Wealth of the fund.

After gathering the data, a team of professionals used a set of linguistic phrases to generate benefit rating information for four mutual funds $S = \{s_1 = extremely\,poor\ benefit, s_2 = very\,poor\ benefit, s_3 = poor\ benefit, s_4 = medium\ benefit, s_5 = good\ benefit, s_6 = very\ good\ benefit, s_7 = extremely\ good\ benefit\}$ of the above five attributes and weight vector of them is $\psi = (0.4, 0.2, 0.1, 0.12, 0.18)^T$, and alternatives Q_1, Q_2, Q_3 and Q_4 evaluated with IVPFULNs by the decision makers have same dominance degree. Evaluation of decision makers is given in Table 1.

Table 1. Evaluations of decision makers

	Q_1	Q_2
G_1	$\langle([s_4, s_5], [.3, .4], [.2, .3], [.1, .2])\rangle$	$\langle([s_4, s_5], [.4, .5], [.1, .2], [.1, .2])\rangle$
G_2	$\langle([s_5, s_5], [.2, .3], [.1, .2], [.2, .3])\rangle$	$\langle([s_5, s_5], [.1, .2], [.3, .4], [.3, .4])\rangle$
G_3	$\langle([s_3, s_4], [.4, .5], [.2, .3], [.1, .2])\rangle$	$\langle([s_4, s_4], [.3, .4], [.1, .2], [.2, .3])\rangle$
G_4	$\langle([s_6, s_6], [.1, .3], [.1, .2], [.2, .3])\rangle$	$\langle([s_5, s_6], [.3, .4], [.2, .3], [.1, .2])\rangle$
G_5	$\langle([s_3, s_4], [.5, .6], [.1, .2], [.1, .2])\rangle$	$\langle([s_4, s_5], [.4, .5], [.1, .2], [.2, .3])\rangle$
	Q_3	Q_4
G_1	$\langle([s_5, s_5], [.2, .3], [.1, .2], [.4, .5])\rangle$	$\langle([s_4, s_5], [.2, .3], [.2, .3], [.3, .4])\rangle$
G_2	$\langle([s_4, s_4], [.4, .5], [.2, .3], [.1, .2])\rangle$	$\langle([s_2, s_3], [.4, .6], [.1, .2], [.1, .2])\rangle$
G_3	$\langle([s_4, s_5], [.1, .3], [.2, .3], [.1, .2])\rangle$	$\langle([s_3, s_6], [.4, .5], [.1, .3], [.1, .2])\rangle$
G_4	$\langle([s_6, s_6], [.3, .5], [.1, .3], [.1, .2])\rangle$	$\langle([s_4, s_5], [.2, .3], [.2, .3], [.3, .4])\rangle$
G_5	$\langle([s_3, s_4], [.4, .5], [.1, .2], [.2, .3])\rangle$	$\langle([s_4, s_4], [.4, .6], [.1, .2], [.1, .2])\rangle$

Case 1:

Step 1. We aggregate IVPFUL information Υ_{ab} for $a = 1, 2, 3, 4; b = 1, 2, 3, 4, 5$ by using IVPFULWA operator to obtain the overall accumulated values Υ_b for $(b = 1, 2, 3, 4)$ represented the alternatives Q_a which is given in the Table 2.

Step 2. Using the aggregated values of the alternatives, which are provided in Table 2, the score values of the alternatives Q_a $(a = 1, 2, 3, 4)$ are displayed below. Then, $\Lambda(\Upsilon_1) = s_{4.498}$, $\Lambda(\Upsilon_2) = s_{4.599}$, $\Lambda(\Upsilon_3) = s_{4.201}$ and $\Lambda(\Upsilon_4) = s_{3.793}$.

Step 3. We create the ranking order of the alternatives as follows based on the values of the scoring function: we obtain $Q_2 \succ Q_1 \succ Q_3 \succ Q_4$. The Q_2 is the best mutual funds for investment.

Find the following outcomes if you use the IVPFULWG operator rather than the IVPFULWA operator.

Table 2. Aggregated values of IVPFULWA operators

$\boldsymbol{Alternative(Q_a)}$	$\boldsymbol{IVPFULWA}$
Q_1	$\langle([s_{4.16}, s_{4.84}], [0.3133, 0.4246], [0.1414, 0.2449], [0.1248, 0.2277])\rangle$
Q_2	$\langle([s_{4.32}, s_{5.02}], [0.3269, 0.4282], [0.1354, 0.2412], [0.1513, 0.2574])\rangle$
Q_3	$\langle([s_{4.46}, s_{4.74}], [0.2859, 0.4084], [0.1231, 0.2371], [0.1972, 0.3104])\rangle$
Q_4	$\langle([s_{3.5}, s_{4.52}], [0.3032, 0.4528], [0.1432, 0.2572], [0.1771, 0.2868])\rangle$

Step 1. We aggregate INUL information Υ_{ab} for $\rho = 1, 2, 3, 4; b = 1, 2, 3, 4, 5$ by using IVPFULWG operator to obtain overall values of Υ_a $(a = 1, 2, 3, 4)$ for Q_a which is given in Table 3.

Step 2. The score for Q_a $(a = 1, 2, 3, 4)$ are shown below by using IVPFULWG operator is given in Table 3. Then, $\Lambda(\Upsilon_1) = s_{4.215}$, $\Lambda(\Upsilon_2) = s_{4.231}$, $\Lambda(\Upsilon_3) = s_{3.768}$ and $\Lambda(\Upsilon_4) = s_{3.415}$.

Step 3. The ranking order of the alternatives is created using the score values of Q_a as follows: $Q_2 \succ Q_1 \succ Q_3 \succ Q_4$. As a result, out of all the funds, Q_2 is still the top mutual fund.

Thus, while the ranking order for Q_a, is unchanged, the best option for operators IVPFULWA (IVPFULWG) is alternative Q_2, which has the highest score of all.

Case 2:

Step 1. we aggregate INUL information Υ_{ab} for $\rho = 1, 2, 3, 4; b = 1, 2, 3, 4, 5$ by using INULDWA operator to obtain the accumulated values of Υ_b for $(a = 1, 2, 3, 4)$ for Q_a which is given in Table 4.

Table 3. Aggregated values of using IVPFULWG operators

$\boldsymbol{Alternative(Q_a)}$	$\boldsymbol{IVPFULWG}$
Q_1	$\langle([s_{4.05}, s_{4.80}], [0.2736, 0.4013], [0.1515, 0.2517], [0.1333, 0.2335])\rangle$
Q_2	$\langle([s_{4.29}, s_{4.99}], [0.2846, 0.3963], [0.1561, 0.2567], [0.1719, 0.2724])\rangle$
Q_3	$\langle([s_{4.36}, s_{4.70}], [0.2549, 0.3873], [0.1312, 0.2436], [0.2508, 0.3529])\rangle$
Q_4	$\langle([s_{3.38}, s_{4.42}], [0.2789, 0.4109], [0.1535, 0.2636], [0.2103, 0.3112])\rangle$

Table 4. Aggregated values of the alternatives using IVPFULDWA operators

$\boldsymbol{Alternative(Q_a)}$	$\boldsymbol{IVPFULDWA}$
Q_1	$\langle([s_{4.16}, s_{4.84}], [0.3250, 0.4362], [0.1333, 0.2400], [0.1190, 0.2239])\rangle$
Q_2	$\langle([s_{4.32}, s_{5.02}], [0.3347, 0.4374], [0.1240, 0.2326], [0.1376, 0.2479])\rangle$
Q_3	$\langle([s_{4.46}, s_{4.74}], [0.2937, 0.4167], [0.1176, 0.2326], [0.1639, 0.2857])\rangle$
Q_4	$\langle([s_{3.5}, s_{4.52}], [0.3103, 0.4717], [0.1351, 0.2521], [0.1531, 0.2703])\rangle$

Step 2. Using the totaled values of the options, the results of Q_a are displayed below in Table 4. Then, $\Lambda(\Upsilon_1) = s_{4.601}$, $\Lambda(\Upsilon_2) = s_{4.740}$, $\Lambda(\Upsilon_3) = s_{4.394}$ and $\Lambda(\Upsilon_4) = s_{3.953}$.

Step 3. Based on computed values of $\Lambda(\Upsilon_a)$, we create the following ranking order for the potential solutions: $Q_2 \succ Q_1 \succ Q_3 \succ Q_4$. Q_2 is the best choice.

The following outcomes are obtained if we employ the IVPFULDWG operator rather than the IVPFULDWA operator.

Step 1. We aggregate IVPFUL data Υ_{ab} by using IVPFULDWG operator to obtain accumulated values of Υ_a for Q_a which is given in Table 5.

Step 2. Using the IVPFULDWG operator, the Q_a score is displayed below in Table 5. Then, $\Lambda(\Upsilon_1) = s_{4.355}$, $\Lambda(\Upsilon_2) = s_{4.384}$, $\Lambda(\Upsilon_3) = s_{3.935}$ and $\Lambda(\Upsilon_4) = s_{3.532}$.

Step 3. The alternatives are ranked in the following order based on the values of the score: $Q_2 \succ Q_1 \succ Q_3 \succ Q_4$. Hence, Q_2 is still the best choice.

According to the calculations above, the two operators IVPFULDWA (IVPFULDWG) have different score values, but the ranking order of the alternatives Q_a, is the same, and the alternative Q_2 is the best option for both operators. As a result, the suggested strategy is reliable within the decision-making framework.

Table 5. Aggregated values of the alternatives using IVPFULDWG operators

$\boldsymbol{Alternative(Q_a)}$	$\boldsymbol{IVPFULDWG}$
Q_1	$\langle([s_{4.05}, s_{4.80}], [0.2523, 0.4087], [0.1383, 0.2274], [0.1194, 0.2077])\rangle$
Q_2	$\langle([s_{4.29}, s_{4.99}], [0.2555, 0.3953], [0.1462, 0.2365], [0.1442, 0.2344])\rangle$
Q_3	$\langle([s_{4.36}, s_{4.70}], [0.2451, 0.4021], [0.1172, 0.2188], [0.2386, 0.3355])\rangle$
Q_4	$\langle([s_{3.38}, s_{4.42}], [0.2833, 0.4087], [0.1404, 0.2400], [0.2039, 0.2966])\rangle$

8 Conclusions

In conclusion, this study presented a methodology utilizing INULNs to address MADM problems. The INULWA, INULWG, INULDWA, and INULDWG operators were introduced, and their properties were investigated. A framework for tackling MADM problems was developed, incorporating these proposed operators. A practical example illustrating the application of the suggested approach for evaluating mutual funds for investment purposes was provided. The proposed model holds potential for application in decision support, cognitive assessment, linguistic research, and various other domains dealing with uncertainty in future studies.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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