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Enhanced Interval State Estimation for Uncertain Systems

Zhaoxia Huang*, Meng Liu, Wanting Dou, Dantong Yang, Xinyu Li, Jiayu Zhang, Ying Wang

Department of mathematics and statistics, Ankang University, 725000 Ankang, China

* Correspondence: Zhaoxia Huang (2007990011@aku.edu.cn)

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Abstract: The quality of state estimation in uncertain systems exerts a significant impact on the performance of control systems. Within these uncertain systems, set-valued mappings introduce output uncertainties, complicating the design of observers. This study maps the output error of uncertain systems to the nonlinear terms of a framer , thereby extending the Luenberger framer. An interval observer design method for uncertain systems is proposed, leveraging monotone system theory to analyze the coherence of the error system. The effectiveness of the algorithm is validated through simulation examples.

Keywords: Uncertain systems; Interval state estimation; Interval observer; Framer

1 Introduction

All practical systems exhibit varying degrees of uncertainty, sometimes manifesting internally and at other times externally. Internal uncertainties such as unmodeled dynamics, unknown system parameters, and unknown control gains are not known a priori to designers. External uncertainties, often presenting as unpredictable or random disturbances, can significantly impact system performance if not adequately considered during control design. Such neglect could prevent the controlled system from achieving desired performance and, in severe cases, might lead to instability. Over recent years, research into uncertain systems has grown, primarily focusing on structural and parametric uncertainties. Around the year 2000, the concept of "interval observers" was formally introduced. The study of Gouzé et al. [1] is one of the earliest journal papers studying interval observers with a significant impact. The study assumes that the uncertainties are unknown but bounded (UBB). Under the premise that the observation error forms a cooperative system [2], simple upper and lower bound state observers of the Luenberger type were designed and successfully applied in biological systems. Although the discussion in this study was limited to linear time-invariant systems and the design conditions were stringent, it undeniably pioneered a new field in observer design. In designing interval observers, the constraint of homogeneity in error systems was removed. It is important to note that traditional asymptotic observers can only provide asymptotic estimates, whereas interval observers are capable of providing bounds on state changes at any given time, essentially composed of both upper and lower bound observers (Figure 1).

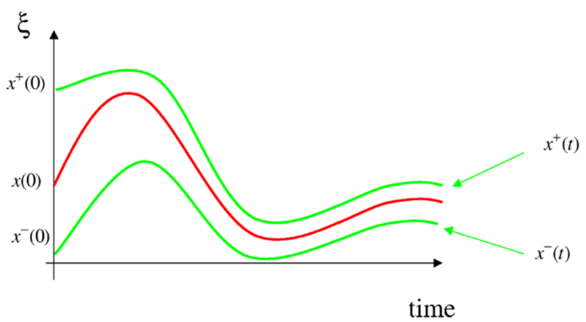

Figure 1. State of the interval observer

Since 2010, interval observers have increasingly garnered the attention of various scholars [3–15], as evidenced by the growing number of papers published annually on the subject in journals such as IEEE Transactions on Automatic Control and Automatica. Unlike the Bernard group, these publications explore interval observer design from various perspectives, expanding from the initial focus on linear time-invariant systems and planar systems to include linear parameter-varying (LPV) systems, chaotic systems [3], feedback linearization systems [7], and Lipschitz nonlinear systems [14]. It has been observed that these various categories of observers are nearly identical in definition. From a methodological standpoint, existing literature on interval observer design can broadly be divided into two categories: designs based on monotone systems theory [16] and those founded on positive systems theory [17].

Representatives of the first category, predominantly led by Bernard and others, utilize the cooperative properties of error systems according to monotone systems theory for designing framers. If the error system is not cooperative but exhibits certain monotonic properties, it can be embedded into a cooperative system using the Mutter theorem [18]. This approach fundamentally extends the comparison principle of differential equations, focusing primarily on the construction of the framer, i.e., function g, followed by an analysis of its stability ($w(t) \equiv 0$) using Lyapunov stability theory. Some existing studies leverage the associated properties of uncertain functions, such as Lipschitz continuity, to construct g, while others employ boundary functions or high-order sliding mode observers. Each method has its unique characteristics, but the discussion primarily targets linear systems or those inherently linear. Extending these methods to time-varying parametrically uncertain nonlinear systems remains an open problem, with current extensions mainly based on LPV systems. Furthermore, the cooperativity of systems is coordinate-dependent, necessitating further in-depth research into the selection of appropriate coordinate transformations.

The second category of methods employs the concept of positive systems to construct two systems as follows:

$$\dot{x}^{+}(t) = \bar{x}\left(x^{+}(t), w^{+}(t), w^{-}(t), y(t)\right)$$

$$\dot{x}^{-}(t) = \underline{x}\left(x^{-}(t), w^{+}(t), w^{-}(t), y(t)\right)$$

This makes the dynamic system concerning the upper and lower bound estimation errors, denoted as $e^{-}(t) = x(t) - x^{-}(t)$ and $e^{+}(t) = x^{+}(t) - x(t)$, is asymptotically stable as a positive system, which ensures:

$$x^{-}\left(x_0^{-}, t\right) \leq x\left(x_0, t\right) \leq x^{+}\left(x_0^{+}, t\right)$$

This approach is comparatively intuitive. Typically, the first category of methods initially designs a stable observer and then applies coordinate transformations to render the error system cooperative; conversely, the second category begins by designing a positive observer and then identifies appropriate gains to stabilize the error system. The tools involved in this method include the internal positive realization theory [18], which boasts the advantage of generally not requiring time-varying coordinate transformations, and permits the consideration of both discrete and continuous cases within the same framework. However, a limitation of this method is its current confinement to linear systems, with the potential extension to nonlinear systems pending further research. Overcoming this limitation depends on advancements in positive systems theory. The estimation of states or the design of observers in uncertain systems has emerged as a sustained area of research interest in recent years. In summary, whether it is based on internal structural parameter estimation or the state estimation reflecting external representations, this research holds significant importance for the study of various stochastic systems. This study primarily delves into an in-depth exploration of state estimation in uncertain systems.

2 Relevant Knowledge

2.1 Transformation of Nonlinear Matrix Inequalities

2.1.1 Transformation of matrix norm constraints

The transformation of matrix norm constraints is outlined below:

$$\|Z(X)\| < 1$$

where, $Z(X) \in R^{n \times q}$, which is equivalent to the following:

$$I_{n \times n} - Z(X)Z^{T}(X) > 0$$

The equivalence is as follows:

$$\begin{bmatrix} I_{n \times n} & Z(X) \\ Z^{T}(X) & I_{n \times n} \end{bmatrix} > 0$$

2.2 Transformation of Nonlinear Weighted Norm Constraints

Nonlinear weighted norm constraints are represented as follows:

$$c^T(X)P^{-1}(X)c(X) < 1$$

where, $c(X) \in R^n, 0 < P(X) \in R^{n\times n}$.

The above expression is formulated as follows in terms of a Linear Matrix Inequality (LMI) problem:

$$\begin{bmatrix} P(X) & c(X) \\ c^T(X) & 1 \end{bmatrix} > 0$$

2.3 Nonlinear Trace Norm Constraints

$$\text{Tr}\left(S^T(X)P^{-1}(X)S(X)\right) < 1$$

where, $S(X) \in R^{n\times q}, 0 < P(X) \in R^{n\times n}$.

By introducing a new variable $Q = Q^T \in R^{p\times p}$, the following conclusion can be drawn:

$$\text{Tr}(Q) < 1, \begin{bmatrix} Q & S^T(X) \\ S(X) & P(X) \end{bmatrix} > 0$$

2.4 Lyapunov Inequality

$$XA + A^TX < 0$$

where, $A \in R^{n\times n}$ is a stable matrix, equivalent to the following LMI:

$$\begin{bmatrix} -XA - A^TX & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix} > 0$$

3 Main Results

The following Lure type differential inclusion system is considered:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Br(t) - G\omega(t) \\ &\omega(t) \in v(Hx(t)) \\ &y(t) = Cx(t) \end{aligned} \tag{1}$$

where, $x(t) \in R^n$ denotes the state of the system, $r(t) \in R^m$ represents the input, $y(t) \in R^r$ denotes the output, $v(\cdot)$ signifies a set-valued mapping, $\omega(t) \in R^q$ represents the output of the set-valued mapping, A, B, C is a real matrix with appropriate dimensions, $H \in R^{q\times n}$ is the input matrix for the set-valued mapping, and $G \in R^{n\times q}$ is the output matrix for the set-valued mapping. Without loss of generality, it is assumed that B and G are of full column rank and C is of full row rank.

If the Luenberger observer exists, then H must necessarily be of full row rank. Conversely, if H is known to be of full row rank, then G is of full column rank. In Lure type differential inclusion systems, it is assumed that the set-valued mappings $v(\cdot)$ are monotonic, meaning that if $\omega_i \in v\left(Hx_i\right), i = 1, 2$, then $\langle \omega_1 - \omega_2, Hx_1 - Hx_2 \rangle \geq 0$.

When considering the output of set-valued mappings, i.e., discussing the following $Cauchy$ problem of differential inclusions:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Br(t) - G\omega(t) \\ &\omega(t) \in v(Hx(t)) \\ &x(0) = x_0 \end{aligned}$$

where, $\omega \in v(\cdot)$.

In the design of differential inclusion observers, the following two methods are employed to address the adaptivity issues:

(i) One approach assumes that for any given differential equation $Cauchy$ issue of $\omega(t) \in v(Hx(t))$, a solution of $\dot{x}(t) = Ax(t) + Br(t) - G\omega(t)$ exists, with $x(0) = x_0$; alternatively, it is assumed that the set-valued mappings $v(\cdot)$ is closed and convex and $Lipschitz$.

(ii) Another method posits that if the set-valued mapping $v(\cdot)$ is maximally monotonic, then not only does a solution of $\dot{x}(t) = Ax(t) + Br(t) - G\omega(t)$ exist, with $x(0) = x_0$, but it is also unique except on a set of measure zero.

Luenberger interval observers are designed for the uncertain systems as follows:

$$\dot{x}^{+} = Ax^{+} + Br(t) + L\left(y - Cx^{+}\right) - G\omega^{+}(t) \tag{2}$$

$$\dot{x}^{-} = Ax^{-} + Br(t) + L\left(y - Cx^{-}\right) - G\omega^{-}(t) \tag{3}$$

$$\omega(t) \in v(Hx(t))$$

where, x is the state, ω is the output of the set-valued mapping, and $L \in R^{n\times r}$, termed the observation gain, functions to ensure that $A - LC$ is a $Hurwitz$ matrix. According to linear system theory, if observability of (A, C) is ensured, then a matrix L must necessarily exist such that $A - LC$ is a $Hurwitz$ matrix.

The difference between systems $\dot{x}^{+} = Ax^{+} + Br(t) + L\left(y - Cx^{+}\right) - G\omega^{+}(t)$ and $\dot{x}(t) = Ax(t) + Br(t) - G\omega(t)$ results in the following:

$$\begin{aligned}
&\dot{x}^{+} - \dot{x} = (A - LC)x^{+} + G\omega(t) - G\omega^{+}(t)\\
&\omega(t) \in v(Hx(t))\\
&\omega^{+}(t) \in v\left(Hx^{+}(t)\right)\\
&y(t) = Cx(t)
\end{aligned}$$

where, $e^{+} = x^{+} - x$ is the observer error.

The above expression can also be expressed as follows:

$$\begin{aligned}
&\dot{e}^{+} = (A - LC)x^{+} + G\left(\omega(t) - \omega^{+}(t)\right)\\
&\omega(t) \in v(Hx(t))\\
&\omega^{+}(t) \in v\left(Hx^{+}(t)\right)
\end{aligned} \tag{4}$$

Similarly, the difference between two systems $\dot{x}(t) = Ax(t) + Br(t) - G\omega(t)$ and $\dot{x}^{-} = Ax^{-} + Br(t) + L\left(y - Cx^{-}\right) - G\omega^{-}(t)$ results in the following:

$$\begin{aligned}
&\dot{x} - \dot{x}^{-} = (A - LC)x^{-} + G\omega^{-}(t) - G\omega(t)\\
&\omega(t) \in v(Hx(t))\\
&\omega^{-}(t) \in v\left(Hx^{-}(t)\right)\\
&y(t) = Cx(t)
\end{aligned}$$

where, $e^{-} = x - x^{-}$ is the observer error.

The above expression can also be expressed as follows:

$$\begin{aligned}
&\dot{e}^{-} = (A - LC)x^{-} + G\left(\omega^{-}(t) - \omega(t)\right)\\
&\omega(t) \in v(Hx(t))\\
&\omega^{-}(t) \in v\left(Hx^{-}(t)\right)
\end{aligned} \tag{5}$$

Theorem 1: It is assumed that $\eta > 0$, a continuous function $V : R^n \to R$ satisfies $V(0) = 0$, and a negative definite continuous function $W : R^n \to R$ exists. These conditions ensure that $D^{+}V(x)(v) \leq W(x)$ satisfies $\|x\| \leq \eta$ and $v \in F(x)$ for all. If for any arbitrarily small $\delta > 0$, then $\|x\| < \delta$ always exists such that $V(x) < 0$. Therefore, the differential inclusion $\dot{x}(t) \in F(x(t))$, with $x(0) = x_0$, is unstable.

Proof: By considering the minimum value of $V(x)$ on $\|x\| \leq \delta$ and the conditions of the theorem, $\min_{\|x\|\delta} V(x) < 0$ can be derived, and the expression is $\arg\min_{\|x\|\leq\delta} V(x)$.

With x_δ as the initial value, the solution of $x(t)$ to $\dot{x}(t) \in F(x(t))$, with $x(0) = x_0$, can be obtained. By considering $x(t)$, the following expression can be obtained according to the conditions of the theorem:

$$D^{+}V(x(t))(1) \leq D^{+}V(x(t))(v)$$

Therefore, from the equation $V(x(t)) - V(x(0)) \leq \int_0^t W(x(s))d_s < 0$, the following can be derived:

$$V(x(t)) - V\left(x_\delta\right) = V(x(t)) - V(x(0)) \leq \int_0^t W(x(s))d_s \leq 0$$

That is, $V(x(t)) \leq V\left(x_\delta\right)$. According to the definition of x_δ, it only holds that $\|x(t)\| \geq \delta$. If $-\lambda = \max_{\delta\leq\|x\|\leq\delta} W(x)$, then $\lambda > 0$.

Similarly to $D^{+}V(x(t))(1) \leq D^{+}V(x(t))(v)$, it can be deduced that:

$$V(x(t)) \leq V\left(x_{\delta}\right)-\lambda t$$

Since $V(x)$ is continuous, it cannot tend towards negative infinity at any finite point x. Therefore, T must exist to ensure $\|x(T)\|=\eta$. It is impossible for the differential inclusion $\dot{x}(t) \in F(x(t))$, with $x(0)=x_0$, to be stable.

Theorem 2: It is assumed that $\eta>0$, a positive definite continuous function $V: R^n \rightarrow R$ satisfies $V(0)=0$ and the $Lipschitz$ condition, and a semidefinite continuous function $W: R^n \rightarrow R$ exists. It is established that for any $x \in \eta B$, there exists $v \in F(x)$ and $D^{-}V(x)(v) \leq W(x)$, such that the differential inclusion $\dot{x}(t) \in F(x(t))$, with $x(0)=x_0$, is weakly stable.

Proof: If it can be demonstrated under the conditions of the theorem that x_0 makes $\delta>0$ satisfy $\|x_0\|<\delta$ for all, and $x(t) \in S_{[0,T]}\left(F, x_0\right)$ exists for any $T>0$ such that the following expression can be established:

$$V(x(t))-V(x(0)) \leq \int_0^t W(x(s)) d_s<0$$

That is,

$$V(x(T))-V\left(x_0\right) \leq \int_0^T W(x(s)) d_s<0$$

Then, it can be understood that $x(t)$ is stable, hence the differential inclusion $\dot{x}(t) \in F(x(t))$, with $x(0)=x_0$, is weakly stable.

Let l and L be the $Lipschitz$ constants of F and V, respectively. Moreover, it is assumed that $x(t)$ is an absolutely continuous function defined on $[0, T]$. The following expression is defined for sufficiently small $\varepsilon>0$:

$$\phi_{\varepsilon}(x(t), t)=V(x(t))-V(x(0))-\int_0^t W(x(s)) d_s-\varepsilon t$$

Then, $\phi_{\varepsilon}(x(t), t)$ is continuous for both $x(t)$ and t, and $\phi_{\varepsilon}(x(0), 0)$. First, the following conclusion is proved:
(i) For any $\varepsilon>0$, there exists $x(t) \in S_{[0,T]}\left(F, x_0\right)$, ensuring that the following expression holds true:

$$\phi_{\varepsilon}(x(t), t)=V(x(t))-V(x(0))-\int_0^t W(x(s)) d_s-\varepsilon t \leq \varepsilon$$

Assuming $x(t)$ is an absolutely continuous function from $[0, T]$ to R^n, then the following expression is defined:

$$\theta(x(t))=\max \left\{t \in[0, T] ; \phi_{\varepsilon}(x(t), t) \leq 0, \max_{s \in[0, t]} \phi_{\varepsilon}(x(s), s) \leq \varepsilon\right\}$$

$\phi_{\varepsilon}(x(0), 0)=0$ is valid for all $x(t)$, hence the set $t \in[0, T] ; \phi_{\varepsilon}(x(t), t) \leq 0, \max_{s \in[0, t]} \phi_{\varepsilon}(x(s), s) \leq \varepsilon$ is non-empty. It is expressed as follows:

$$\hat{t}=\sup \in\left\{\theta(x(\cdot)) ; x(\cdot) \in S_{[0,T]}\left(F, x_0\right)\right\}$$

The following two facts are proven:
(a) The existence of $\hat{x}(\cdot) \in S_{[0,T]}\left(F, x_0\right)$ leads to $t=\theta(\hat{x})$.

According to the definition of $\hat{t}$, the existence of $x_k(\cdot) \in S_{[0,T]}\left(F, x_0\right)$ ensures that $\hat{t}_k=\sup \left\{\theta\left(x_k(\cdot)\right)\right\}$ satisfies $t_k \uparrow \hat{t}$. Since $F(x)$ is bounded, $\{x_k(t)\}$ is equicontinuous. A convergent subsequence exists, it can be assumed that $\{x_k(t)\}$ is converging, whose limit is $\hat{x}(t)$. Since $\phi_{\varepsilon}(x(t), t)$ is continuous at for $x(t)$, it is concluded that $\hat{t}=\theta(\hat{x})$.

(b) $\hat{t}=T$.

It is assumed that $\hat{t}<T$, then $\phi_{\varepsilon}(\hat{x}(\hat{t}), \hat{t})=0$. Under the conditions of the theorem, the existence of $v \in F(\hat{x}(\hat{t}))$ for $\hat{x}(\hat{t})$ leads to the following expression:

$$D^{-}V(\hat{x}(\hat{t}))(v) \leq W(\hat{x}(\hat{t}))$$

Therefore, the following expression holds:

$$\begin{aligned}
D^{-}\phi_{\varepsilon}(\hat{x}(\hat{t}), t)(v, 1) &= D^{-}\left[V(\hat{x}(\hat{t}))-V\left(x_0\right)-\int_0^{\hat{t}} W(x(s)) d_s-\hat{\varepsilon t}\right](v, 1) \\
&= D^{-}V(\hat{x}(\hat{t}))(v)-W(x(\hat{t}))-\varepsilon \\
&\leq -\varepsilon
\end{aligned}$$

The function is thus defined as follows:

$$y(t) = \begin{cases} \hat{x}(t), t < \hat{t}, \\ \hat{x}(\hat{t}) + (t - \hat{t})v, t \geq \hat{t} \end{cases}$$

By considering the following expression when $t \geq \hat{t}$:

$$\begin{aligned} d(\dot{y}(t), F(y(t))) &= d(v, F(\hat{x}(\hat{t})) + (t - \hat{t})v) \\ &\leq d(F(\hat{x}(\hat{t}),) + F(\hat{x}(\hat{t}),) + (t - \hat{t})v) \\ &\leq l\|v\|\|t - \hat{t}\| \end{aligned}$$

Then the existence of $x(t) \in S_{[\hat{t},T]}(F, \hat{x}(\hat{t}))$ leads to the establishment of the following expression:

$$\|y(t) - x(t)\| \leq \int_{\hat{t}}^{t} e^{l(t-s)} l\|v\|(s - \hat{t}) d_s$$

For convenience of notation, the following expression is defined:

$$\alpha(t - \hat{t}) = \int_{\hat{t}}^{t} e^{l(t-s)} l\|v\|(s - \hat{t}) d_s$$

It can then be easily demonstrated that the following expression holds true.

$$\begin{aligned} &\lim_{t \uparrow \hat{t}} \frac{\alpha(t - \hat{t})}{t - \hat{t}} = 0 \\ &\phi_\varepsilon(x(t), t) - \phi_\varepsilon(y(t), t) = V(x(t)) - V(y(t)) \end{aligned}$$

When $t \in [\hat{t}, T]$, it leads to the following expression:

$$\phi_\varepsilon(x(t), t) \leq \phi_\varepsilon(y(t), t) + L\alpha(t - \hat{t})$$

Based on the following equation:

$$\begin{aligned} D^- \phi_\varepsilon(\hat{x}(\hat{t}), t)(v, 1) &= D^- \left[V(\hat{x}(\hat{t})) - V(x_0) - \int_0^{\hat{t}} W(x(s)) d_s - \varepsilon \hat{t} \right] (v, 1) \\ &= D^- V(\hat{x}(\hat{t}))(v) - W(x(\hat{t})) - \varepsilon \\ &\leq -\varepsilon \end{aligned}$$

The following can be derived:

$$\liminf_{t \downarrow \hat{t}} \frac{\phi_\varepsilon(\hat{x}(\hat{t}) + (t - \hat{t})v, t) - \phi_\varepsilon(\hat{x}(\hat{t}), \hat{t})}{t - \hat{t}} \leq -\varepsilon$$

Since $\phi_\varepsilon(\hat{x}(\hat{t}), \hat{t}) = 0$, there must necessarily exist $\tilde{t} \geq \hat{t}$ such that the following expressions can be established:

$$\phi_\varepsilon(y(\tilde{t}), \tilde{t}) = \phi_\varepsilon(\hat{x}(\hat{t}) + (\tilde{t} - \hat{t})v, \tilde{t}) \leq -\frac{\varepsilon}{2}(\tilde{t} - \hat{t})$$

$$\alpha(t - \hat{t}) \leq \frac{\varepsilon}{2L}(t - \hat{t}), t \in [\hat{t}, \tilde{t}]$$

It can be noted that the preceding expression is only valid at the moment $t = \tilde{t}$. By applying the following expression:

$$\phi_\varepsilon(x(t), t) \leq \phi_\varepsilon(y(t), t) + L\alpha(t - \hat{t})$$

Initially, it can be derived:

$$\phi_\varepsilon(x(\tilde{t}), \tilde{t}) \leq 0$$

Subsequently, it can be derived:

$$\begin{aligned}\phi_{\varepsilon}(x(t),t) &\leq \phi_{\varepsilon}(y(t),t)+L\alpha(t-\hat{t})\\ &=\phi_{\varepsilon}(\hat{x}(\hat{t})+(t-\hat{t})v,t)+L\alpha(t-\hat{t})\\ &\leq \phi_{\varepsilon}(\hat{x}(\hat{t}))+L\|v\|(t-\hat{t})+L\alpha(t-\hat{t})\\ &=L\|v\|(t-\hat{t})+L\alpha(t-\hat{t})\end{aligned}$$

Clearly, when $t-\hat{t}$ is sufficiently small, it is possible to ensure that $\phi_{\varepsilon}(x(t),t)\leq\varepsilon$. Thus, by employing this $x(t)$, it can be ensured that $\theta(x(t))>\hat{t}$, which contradicts the definition of $\hat{t}$. Therefore, it can be concluded that $\hat{t}=T$.

(ii) The proof is provided that the following expression holds:

$$V(x(t))-V(x(0))\leq\int_0^t W(x(s))d_s<0$$

A sequence ε_n is selected, with $\varepsilon_n\downarrow 0$. For each ε_n, according to the proof of (i), there exists $x_n(t)\in S_{[0,T]}(F,x_0)$ such that $\phi_{\varepsilon}(x_n(t),t)\leq\varepsilon_n$. Since $\{x_n(t)\}$ contains a convergent subsequence, it can be assumed that $x_n(t)\to x(t)$, then $x(t)\in S_{[0,T]}(F,x_0)$.

According to the following definition:

$$\phi_{\varepsilon}(x(t),t)=V(x(t))-V(x(0))-\int_0^t W(x(s))d_s-\varepsilon t$$

The following expression can be obtained:

$$V(x(t))-V(x(0))\leq\int_0^t W(x(s))d_s<0$$

Thus, the theorem is proven.

4 Simulation

By considering the following uncertain systems:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-10&-3&-1\\ 6&-5&4\\ 1&0&-9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}-\begin{bmatrix}1\\ 2\\ 2\end{bmatrix}\omega+\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}r$$
$$y=x_1$$

where, the set-valued mapping $v(\cdot)$ is as follows:

$$v(x_1+3x_2+2x_3)=\begin{cases}x_1+3x_2+2x_3+3\operatorname{sgn}(x_1+3x_2+2x_3), & x_1+3x_2+2x_3\neq 0\\ [-3,3], & x_1+3x_2+2x_3=0\end{cases}$$

$$\omega_1(t)=\begin{cases}-1 & x_1+x_2<-2\\ 0.5(x_1+x_2) & -2\leq x_1+x_2<0\\ 0.5(x_1+x_2) & 0\leq x_1+x_2<2\\ 1 & \leq 2x_1+x_2\end{cases}$$

$$\omega_2(t)=\begin{cases}-1 & x_1+x_2<-2\\ -1 & -2\leq x_1+x_2<0\\ x_1+x_2-1 & 0\leq x_1+x_2<2\\ 1 & \leq 2x_1+x_2\end{cases}$$

In this system, $A=\begin{bmatrix}-10&-3&-1\\ 6&-5&4\\ 1&0&-9\end{bmatrix}$, $B=\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$, $C=\begin{bmatrix}1&0&0\end{bmatrix}$, and $G=\begin{bmatrix}1\\ 2\\ 2\end{bmatrix}$.

Selecting $L=\begin{bmatrix}-6&2&1\end{bmatrix}^T$ leads to the following expression:

$$A-LC=\begin{bmatrix}-4&-3&-1\\ 4&-5&4\\ 0&0&-9\end{bmatrix}$$

It can be calculated that $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$, and $H = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$.

Figure 1 illustrates the interval state estimation of the system, revealing that the following expression exists at any given moment:

$$x_i^-(t) \leq x_i(t) \leq x_i^+(t), \quad i = 1, 2, 3$$

The proposed interval observers (2) and (3) provide an effective interval estimation of the system state.

Figure 2 demonstrates that under the selected set-valued mapping, both the uncertain systems and their observer systems are asymptotically stable, with the observer system trajectories tracking the original state trajectories at any given time.

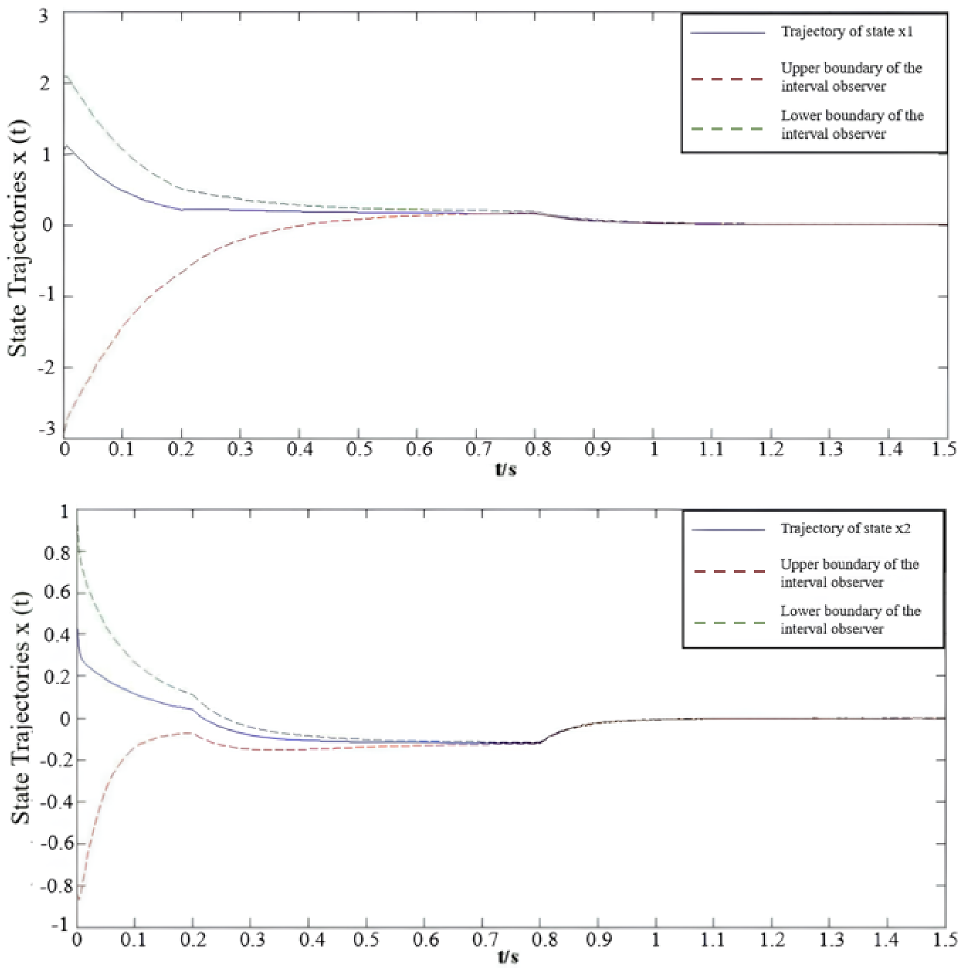

Figure 2. The state trajectories of Lure systems and interval observers

5 Conclusions

This study has investigated the design of interval observers for uncertain systems under different choices of set-valued mappings. Due to the inherent uncertainty in the outputs of uncertain systems caused by set-valued mappings, designing observers poses significant challenges. In this study, the output errors of the system were mapped onto the nonlinear terms of a framer, thereby developing an extended Luenberger framer. Based on monotone systems theory, the coherence of the error system was analyzed, and a design method for interval observers for uncertain systems was proposed. It was demonstrated that both the uncertain system and its observer system are asymptotically stable. Additionally, the observer system's trajectories were shown to track the original state trajectories at any given moment through algorithmic simulation under the selection of set-valued mappings.

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Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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