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Addressing Cost-Efficiency Problems Based on Linear Ordering of Piecewise Quadratic Fuzzy Quotients

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Abstract: Ordering of quotients is a critical aspect of cost-efficiency problems, which hold significant interest and importance for suppliers of goods and services as well as consumers. Comparisons (ordering) are straightforward when dealing with ordinary numbers, yet in many instances, the data are imprecise, vague, or subject to seasonal variations. Consequently, such data may be unknown or derive from expert opinions. Unlike ordinary numbers, fuzzy data render quotients only partially ordered. This study examines the linear ordering of quotients with fuzzy data, expressed in terms of confidence intervals, α-cuts, or piecewise quadratic fuzzy numbers (PQFNs), within the context of cost-efficiency problems. Moreover, the challenges associated with quotient ordering in cost-efficiency problems are introduced.

Keywords: Cost efficiency; Linear ordering; Fuzzy sets; α-level sets; Piecewise quadratic fuzzy numbers; Confidence intervals; Distance between fuzzy numbers

1 Introduction

Cost-efficiency is a crucial criterion in business management, representing a strategy for optimizing the balance between cost expenditure and product design or activity execution. In recent years, decision-makers have increasingly adopted the concept of cost-efficiency to navigate uncertain model parameters.

Consequently, as known, fuzzy set theory was introduced by Zadeh [1] to deal with fuzziness. Up to now, fuzzy set theory has been applied to broad fields. Fuzzy numerical data can be represented by means of fuzzy subsets of the real line, known as fuzzy numbers. For the fuzzy set theory development, we may referee to the papers of Kaufmann and Gupta [2], and Dubois and Prade [3], they extended the use of algebraic operations of real numbers to fuzzy numbers by the use a fuzzifaction principle. Bellman and Zadeh [4] introduced the concept of a maximizing decision making problem. Subsequently, fuzzy sets have been extensively studied and applied in various domains such as polynomial form fuzzy numbers [5], transportation problems [6], and critical path activity networks [7].

In the literature, interval numbers have been considered to address the uncertainty inherent in model parameters. A fuzzy number is a reference to a fuzzy interval, which is understood as an interval extension of the fuzzy number. Numerous authors have studied interval numbers, including Moore [8], Grzegorzewski [9], Abbasbandy and Amirfakhrian [10, 11].

In the business context, cost-efficiency is typically measured by monitoring the output-to-cost ratio. Liu et al. [12] presented a cost-efficiency model based on emergency resources, employing a multi-objective programming problem for evaluation. Various cost-efficiency applications have been explored using the fuzzy concept, such as Payan and Hekmatnia [13], Dehnokhalaji et al. [14], and Kumar [15], who determined total costs in an inventory management problem. Chung et al. [16] investigated the optimization of cost-efficiency using an electric vehicle charge scheduling approach. More recently, Nazila and Samira [17] presented work on generalized fuzzy cost-efficiency problems.

In other instances, cost-efficiency determination involves measuring the revenue generated against expenses incurred. Yang et al. [18] examined fuzzy programming with nonlinear membership functions using piecewise linear approximation. Several researchers have utilized piecewise linear membership functions, including Effati and

Abbasiyan [19] for fuzzy linear programming and Coroianu et al. [20] for linear approximation of fuzzy numbers. Sen and Pal [21] investigated a fuzzy separable quadratic programming model using piecewise linear approximation. Chalco-Cano et al. [22] introduced the concept of generalized convexity of fuzzy mappings through linear ordering, while Nayagam et al. [23] studied linear ordering for trapezoidal intuitionistic fuzzy numbers. Other applications of piecewise quadratic fuzzy numbers have been explored by Gong et al. [24] and Khalifa and Kumar [25, 26] in the context of discounting problems.

This study focuses on cost-efficiency problems involving linear ordering of quotients with fuzzy data expressed in terms of closed interval approximation, α-cuts, or piecewise quadratic fuzzy numbers (PQFNs).

The remainder of the paper is organized as follows: Section 2 introduces some preliminaries needed in this paper. Section 3 presents quotients with intervals of confidence, Section 4 presents quotients of α-cuts. Finally, concluding remarks are reported in Section 5.

2 Preliminaries

In order to easily discuss the problem, it recalls basic rules and findings related to fuzzy numbers, piecewise quadratic fuzzy numbers, close interval approximation and its arithmetic operations.

Definition 1 [27]. Fuzzy number: A fuzzy number $\tilde{A}$ is a fuzzy set with a membership function defined as $\pi_{\tilde{A}}(\mathrm{x}) : \Re \to [0, 1]$, and satisfies:

1. $\tilde{\mathrm{A}}$ is fuzzy convex, i.e., $\pi_{\tilde{A}}(\delta \mathrm{x} + (1-\delta)\mathrm{y}) \geq \min\{\pi_{\tilde{A}}(\mathrm{x}), \pi_{\tilde{A}}(\mathrm{y})\}\,; \forall \mathrm{x}, \mathrm{y} \in \Re\,; 0 \leq \delta \leq 1$;
2. $\widetilde{\mathrm{A}}$ is normal, i.e., $\exists \mathrm{x}_0 \in \Re$ for which $\pi_{\tilde{A}}(\mathrm{x}_0) = 1$;
3. $\mathrm{Supp}(\tilde{A}) = \{\mathrm{x} \in \Re : \pi_{\tilde{A}}(\mathrm{x}) > 0\}$ is the support of $\tilde{\mathrm{A}}$;
4. $\pi_{\tilde{A}}(\mathrm{x})$ is an upper semi- continuous (i.e., for each $\alpha \in (0, 1)$, the α - cut set $\tilde{A}_\alpha = \{x \in \Re : \pi_{\tilde{A}} \geq \alpha\}$ is closed.

Definition 2 [28]. A piecewise quadratic fuzzy number (PQFN) is denoted by $\widetilde{A}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ are real numbers, and is defined by if its membership function $\mu_{\tilde{\mathrm{a}}_{\mathrm{PQ}}}$ is given by (see Figure 1)

$$\mu_{\tilde{A}_{PQ}} = \begin{cases} 0, & x < a_1, \\ \frac{1}{2}\frac{1}{(a_2-a_1)^2}(x-a_1)^2, & a_1 \leq x \leq a_2, \\ \frac{1}{2}\frac{1}{(a_3-a_2)^2}(x-a_3)^2+1, & a_2 \leq x \leq a_3, \\ \frac{1}{2}\frac{1}{(a_4-a_3)^2}(x-a_3)^2+1, & a_3 \leq x \leq a_4, \\ \frac{1}{2}\frac{1}{(a_5-a_4)^2}(x-a_5)^2, & a_4 \leq x \leq a_5, \\ 0, & x > a_5. \end{cases}$$

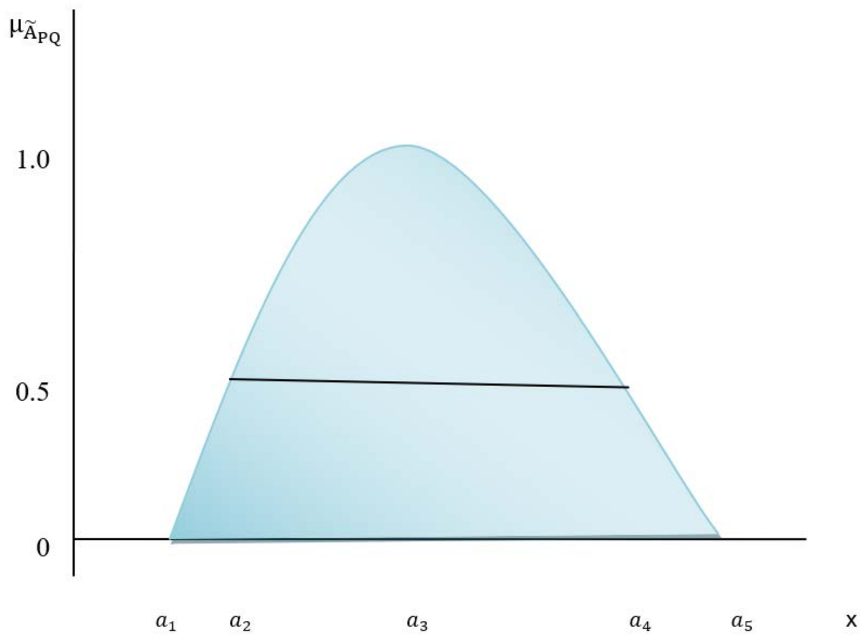

Figure 1. Graph illustration of a PQFN

The interval of confidence at level α for the PQFN is defined as $\left(\tilde{A}_{PQ}\right)_{\alpha} = [a_1 + 2(a_2 - a_1)\alpha, a_5 - 2(a_5 - a_4)\alpha]; \forall\alpha \in [0,1]$.

Definition 3 [28]. Let $\widetilde{A}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$ and $\widetilde{B}_{PQ} = (b_1, b_2, b_3, b_4, b_5)$ be two piecewise quadratic fuzzy numbers. The arithmetic operations on $\widetilde{\mathrm{A}}_{\mathrm{PQ}}$ and $\widetilde{\mathrm{B}}_{\mathrm{PQ}}$ are:

(i) Addition: $\tilde{A}_{PQ}(+)\tilde{B}_{PQ} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$.

(ii) Subtraction: $\widetilde{\mathrm{A}}_{\mathrm{PQ}}(-)\widetilde{\mathrm{B}}_{\mathrm{PQ}} = (\mathrm{a}_1 + \mathrm{b}_5, \mathrm{a}_2 + \mathrm{b}_4, \mathrm{a}_3 + \mathrm{b}_3, \mathrm{a}_4 + \mathrm{b}_2, \mathrm{a}_5 + \mathrm{b}_1)$.

(iii) Scalar multiplication: $k\tilde{A}_{PQ} = \begin{cases} (ka_1, ka_2, ka_3, ka_4, ka_5), k > 0, \\ (ka_5, ka_4, ka_3, ka_2, ka_1), k < 0. \end{cases}$

Definition 4 [28]. An interval approximation $[A] = [a_\alpha^-, a_\alpha^+]$of a PQFN $\tilde{A}$ is called closed interval approximation if: $a_\alpha^- = \inf\{x \in \mathbb{R} : \mu_{\tilde{A}} \geq 0.5\}$, and $a_\alpha^+ = \sup\{x \in \mathbb{R} : \mu_{\tilde{A}} \geq 0.5\}$.

Definition 5. Associated ordinary number [28]. If $[A] = [a_\alpha^-, a_\alpha^+]$ is the close interval approximation of PQFN, the Associated ordinary number of [A] is defined as $\hat{A} = \frac{a_\alpha + a_\alpha^+}{2}$.

Definition 6 [28]. Let $[A] = [a_\alpha^-, a_\alpha^+]$, and $[B] = [b_\alpha^-, b_\alpha^+]$ be two interval approximations of PQFN. Then the arithmetic operations are:

1. Addition: $[A](+)[B] = [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+]$,
2. Subtraction: $[A](-)[B] = [a_\alpha^- - b_\alpha^+, a_\alpha^+ - b_\alpha^-]$,
3. Scalar multiplication: $\alpha[A] = \begin{cases} [\alpha a_\alpha^-, \alpha a_\alpha^+], \alpha > 0 \\ [\alpha a_\alpha^+, \alpha a_\alpha^-], \alpha < 0 \end{cases}$,
4. Multiplication: $[A](\times)[B] = \left[\frac{a_\alpha^+ b_\alpha^- + a_\alpha^- b_\alpha^+}{2}, \frac{a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+}{2}\right]$,
5. Division: $[A](\div)[B] = \begin{cases} \left[2\left(\frac{a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right), 2\left(\frac{a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right)\right], [B] > 0, b_\alpha^- + b_\alpha^+ \neq 0 \\ \left[2\left(\frac{a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right), 2\left(\frac{a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right)\right], [B] < 0, b_\alpha^- + b_\alpha^+ \neq 0 \end{cases}$.

•Notation

In this section, some of notation needed in the paper are used:

$N[N_1, N_2]$: Numerator

$D[D_1, D_2]$: Denominator

$Q[Q_1, Q_2]$: Quotient, $N(:)D$

$\hat{A}$: Associated ordinary number

$D(\tilde{A}, \tilde{B})$: Distance between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

3 Quotients with Interval of Confidence

In order to begin with interval of approximations, we shall assume that all the data under consideration are positive (i.e., that is, that they are elements of $\Re_0^+$).

A numerator N and denominator D will be represented by an interval confidence as follows:

$$N = [N_1, N_2] \text{ and } D = [D_1, D_2].$$

Their quotient is, therefore,

$$Q = N(:)D = [N_1, N_2](:)[D_1, D_2] = \left[\frac{N_1}{D_2}, \frac{N_2}{D_1}\right], \text{ or } [Q_1, Q_2] = \left[\frac{N_1}{D_2}, \frac{N_2}{D_1}\right] \tag{1}$$

We shall now give a procedure to compare the quotients. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals of confidence in $\Re_0^+$. Their upper bound is given by

$$A(\vee)B = [a_1, a_2](\vee)[b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2] \tag{2}$$

and their lower bound by

$$A(\wedge)B = [a_1, a_2](\wedge)[b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2] \tag{3}$$

To order A and B, we can use the sum of left and right distances,

$$D(A, A(\vee)B) = |a_1 - a_1 \vee b_1| + |a_2 - a_2 \vee b_2| \tag{4}$$

$$D(B, A(\vee)B) = |b_1 - a_1 \vee b_1| + |b_2 - a_2 \vee b_2| \tag{5}$$

By convention, if $D(A, A(\vee)B) < D(B, A(\vee)B)$, then A will be preferred to B. If both distances are equal, they will be indifferent.

Instead of using the upper bound, the lower bound can also be considered. Thus,

$$D(A, A(\wedge)B) = |a_1 - a_1 \wedge b_1| + |a_2 - a_2 \wedge b_2| \tag{6}$$

$$D(B, A(\wedge)B) = |b_1 - a_1 \wedge b_1| + |b_2 - a_2 \wedge b_2| \tag{7}$$

By convention, if $D(A, A(\wedge)B) > D(B, A(\wedge\vee)B)$, then A will be preferred to B. If both distances are equal, they will be indifferent.

The symbol $>$ corresponds to the research of the maximum. This preference will be reversed if we use $<$ instead of $>$ and in this case, this corresponds to the research of the minimum.

The proceeding formulas can be generalized for m intervals of confidence. The generalization is given by

$$\begin{aligned}
&(\vee_{i=1}^{m} A_i) = [\vee_{i=1}^{m} a_{1i}, \vee_{i=1}^{m} a_{2i}]\,,\\
&(\wedge_{i=1}^{m} A_i) = [\wedge_{i=1}^{m} a_{1i}, \wedge_{i=1}^{m} a_{2i}]\,,\\
&D\left(A_i, (\vee_{i=1}^{m} A_i)\right) = |a_{1i} - \vee_{i=1}^{m} a_{1i}| + |a_{2i} - \vee_{i=1}^{m} a_{2i}|\,,\\
&D\left(A_i, (\wedge_{i=1}^{m} A_i)\right) = |a_{1i} - \wedge_{i=1}^{m} a_{1i}| + |a_{2i} - \wedge_{i=1}^{m} a_{2i}|\,.
\end{aligned} \tag{8}$$

Because D is a distance (a scalar), it introduces a linear order (strict or non-strict). In the non- strict case, equivalent classes are introduced. We shall now apply the foregoing discussion on linear- ordering to quotient problem.

$$Q^{(1)}(\vee)Q^{(2)} = \left[\frac{N_1^{(1)}}{D_2^{(1)}}, \frac{N_2^{(1)}}{D_1^{(1)}}\right] \vee \left[\frac{N_1^{(2)}}{D_2^{(2)}}, \frac{N_2^{(2)}}{D_1^{(2)}}\right] = \left[\frac{N_1^{(1)}}{D_2^{(1)}} \vee \frac{N_1^{(2)}}{D_2^{(2)}}, \frac{N_2^{(1)}}{D_1^{(1)}} \vee \frac{N_2^{(2)}}{D_1^{(2)}}\right] \tag{9}$$

and

$$Q^{(1)}(\wedge\vee)Q^{(2)} = \left[\frac{N_1^{(1)}}{D_2^{(1)}}, \frac{N_2^{(1)}}{D_1^{(1)}}\right] \wedge \left[\frac{N_1^{(2)}}{D_2^{(2)}}, \frac{N_2^{(2)}}{D_1^{(2)}}\right] = \left[\frac{N_1^{(1)}}{D_2^{(1)}} \wedge \frac{N_1^{(2)}}{D_2^{(2)}}, \frac{N_2^{(1)}}{D_1^{(1)}} \wedge \frac{N_2^{(2)}}{D_1^{(2)}}\right] \tag{10}$$

If we have n intervals of confidence then using (9), the distances can be computed in the same way.

Example 1

In this example, we consider a case of three intervals of confidence with numerators, denominators and their quotient given by

$$\begin{aligned}
N^{(1)} = [5, 11], D^{(1)} = [13, 17], Q^{(1)} = [0.294, 0.846]\\
N^{(2)} = [5, 11], D^{(2)} = [13, 17], Q^{(2)} = [0.173, 0.888]\\
N^{(3)} = [3, 13], D^{(3)} = [5, 11], Q^{(3)} = [0.272, 2.600].
\end{aligned}$$

We look for the upper bound of the quotient and find

$$Q^{(1)}(\vee), Q^{(2)}(\vee)Q^{(3)} = [0.294, 0.846](\vee)[0.173, 0.888](\vee)[0.272, 2.600] = [0.294, 2.600].$$

We compute now the respective distances

$$\begin{aligned}
D\left(Q^{(1)}, Q^{(1)}(\vee), Q^{(2)}(\vee)Q^{(3)}\right) = |0.294 - 0.294| + |0.846 - 2.600| = 0 + 1.754 = 1.754,\\
D\left(Q^{(2)}, Q^{(1)}(\vee), Q^{(2)}(\vee)Q^{(3)}\right) = |0.173 - 0.294| + |0.888 - 2.600| = 0.121 + 1.712 = 1.833,\\
D\left(Q^{(3)}, Q^{(1)}(\vee), Q^{(2)}(\vee)Q^{(3)}\right) = |0.181 - 0.294| + |2.600 - 2.600| = 0.133 + 0 = 0.133.
\end{aligned}$$

From this we obtain a linear order of the three quotients as $Q^{(1)} > Q^{(2)} > Q^{(3)}$ because $Q^{(3)}$ is the nearest with regard to the upper bound after $Q^{(1)}$ and $Q^{(2)}$.

4 Quotient with α-Cuts

We shall now extend our discussion of linear ordering to fuzzy numbers using α-cuts.

Let $N_1(\alpha), N_2(\alpha), D_1(\alpha), D_2(\alpha) \in \Re_0^+$, and $\forall\alpha \in [0, 1]$. Then, we have

$$\begin{aligned}
&N(\alpha) = [N_1(\alpha), N_2(\alpha)]\,, D(\alpha) = [D_1(\alpha), D_2(\alpha)]\,,\\
&Q(\alpha) = [Q_1(\alpha), Q_2(\alpha)] = \left[\frac{N_1(\alpha)}{D_2(\alpha)}, \frac{N_2(\alpha)}{D_1(\alpha)}\right]
\end{aligned} \tag{11}$$

Recall that the upper bound of the α-cuts is given by

$$A^*(\alpha) = \left(\vee_{i=1}^{n}\right) A_i(\alpha) = \left[\vee_{i=1}^{n} a_{1i}(\alpha), \vee_{i=1}^{n} a_{2i}(\alpha)\right] = \left[a_1^*(\alpha), a_2^*(\alpha)\right]. \tag{12}$$

And that the lower bound of the α-cuts is

$$A^*(\alpha) = \left(\wedge_{i=1}^{n}\right) A_i(\alpha) = \left[\wedge_{i=1}^{n} a_{1i}(\alpha), \wedge_{i=1}^{n} a_{2i}(\alpha)\right] = \left[a_{*_1}(\alpha), a_{*_2}(\alpha)\right]. \tag{13}$$

The distances of the α-cuts are given by
For the upper bound:

$$D\left(A_i(\alpha), A^*(\alpha)\right) = \int_{\alpha=0}^{1} \left|a_{1i}(\alpha) - a_1^*(\alpha)\right| d\alpha + \int_{\alpha=0}^{1} \left|a_{2i}(\alpha) - a_2^*(\alpha)\right| d\alpha, i = 1, \ldots, n. \tag{14}$$

And for the lower bound:

$$D\left(A_i(\alpha), A_*(\alpha)\right) = \int_{\alpha=0}^{1} \left|a_{1i}(\alpha) - a_{*_1}(\alpha)\right| d\alpha + \int_{\alpha=0}^{1} \left|a_{2i}(\alpha) - a_{*_2}(\alpha)\right| d\alpha, i = 1, \ldots, n. \tag{15}$$

These formulas will be applied now to the quotient problems with α-cuts

$$Q_i(\alpha) = \left[\frac{N_{1i}(\alpha)}{D_{2i}(\alpha)}, \frac{N_{2i}(\alpha)}{D_{1i}(\alpha)}\right], i = 1, \ldots, n. \tag{16}$$

Example 2
Consider the following three fuzzy quotients

$$Q_1 = N_1(:)D_1, Q_2 = N_2(:)D_2, Q_3 = N_3(:)D_3$$

where the numerical values are given in α-cuts for $\alpha = 0, 0.1, \ldots 0.9, 1$ as shown in Table 1, Table 2 and Table 3.

Table 1. Fuzzy quotient $Q_1(\alpha)$, Example 2

α	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$N_1(\alpha)$	[7,11]	[7,11]	[7,10]	[7,10]	[8,10]	[8,10]	[8,10]	[9,10]	[9,10]	9	9
$D_1(\alpha)$	[4,13]	[4,12]	[4,12]	[5,12]	[5,11]	[5,11]	[5,10]	[6,10]	[8,9]	[8,9]	8
$Q_1(\alpha)$	[0.538,2.75]	[0.583,2.75]	[0.583,2.5]	[0.583,2]	[0.727,2]	[0.727,2]	[0.8,2]	[0.9,1.7]	[1,1.250]	[1,1.125]	1.125

Table 2. Fuzzy quotient $Q_2(\alpha)$, Example 2

α	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$N_2(\alpha)$	[9,18]	[9,17]	[9,17]	[10,16]	[10,15]	[10,15]	[10,14]	[10,13]	[11,13]	[11,12]	12
$D_2(\alpha)$	[4,13]	[4,13]	[4,12]	[5,11]	[5,11]	[6,10]	[6,10]	[6,10]	[7,9]	[8,9]	8
$Q_2(\alpha)$	[0.692,4.5]	[0.692,2.75]	[0.583,4.25]	[0.750,4.25]	[0.909,3.2]	[1,2.5]	[1,2.33]	[1,2.17]	[1.22,1.86]	[1.22.1.5]	1.5

Table 3. Fuzzy quotient $Q_3(\alpha)$, Example 2

α	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$N_3(\alpha)$	[1,20]	[2,19]	[3,17]	[4,15]	[6,13]	[7,12]	[8,12]	[8,12]	[10,12]	11	11
$D_3(\alpha)$	[3,18]	[3,8]	[4,8]	[4,8]	[4,8]	[5,7]	[5,7]	[5,7]	[6,7]	[6,7]	7
$Q_3(\alpha)$	[0.125,6.666]	[0.250,6.33]	[0.375,4.25]	[0.5,3.75]	[0.75,3.25]	[1,2.4]	[1.14,2.4]	[1.14,2.4]	[1.4,2]	[1.571,1.83]	1.571

The first step is to find the upper bound $Q_1(\vee)Q_2(\vee)Q_3$ which is given in Table 4.

Table 4. Upper bound of the three fuzzy quotients given in Table 1, Table 2 and Table 3

α	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$\vee_{i=1}^{3} Q_i(\alpha)$	[0.692,6.666]	[0.692,6.333]	[0.750,4.250]	[0.909,3.750]	[0.909,3.250]	[1,2.5]	[1.142,2.4]	[1.142,2.4]	[1.43,2]	[1.571,1.83]	1.571

The second step is the computation of the distances which are given in Table 5, Table 6 and Table 7.
The total order of distance is 2.052<6.807<19.027.
Therefore, the ordering of the preference is $Q_3 > Q_2 > Q_1$.

Table 5. Computation of distance for fuzzy quotient $Q_1(\alpha)$, Example 2

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Total
$D(Q_1, \vee Q_i)$	4.070	3.692	1.917	2.076	1.432	0.773	0.742	0.976	1.178	1.279	0.982	19.027

Table 6. Computation of distance for fuzzy quotient $Q_2(\alpha)$, Example 2

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Total
$D(Q_2, \vee Q_i)$	2.166	2.083	0	0.550	0.250	0	0.209	0.376	0.349	0.682	0.142	6.807

Table 7. Computation of distance for fuzzy quotient $Q_3(\alpha)$, Example 2

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Total
$D(Q_3, \vee Q_i)$	0.567	0.442	0.375	0.409	0.249	0.1	0	0	0	0	0	2.052

5 Conclusions and Future Works

In this paper, we have provided a solution of linear ordering of fuzzy quotients for two different cases: quotients with interval of confidence, and quotients with α-cuts. Such problems are often encountered in cost efficiency and other related areas. In the Future work might contain the additional extension of this study to other fuzzy-like structure (i.e., Neutrosophic set, interval-valued fuzzy set, Spherical fuzzy set, Pythagorean fuzzy set etc. In addition, one can consider new fuzzy systems such as interval type-2, interval type-3, Possibility Interval-valued Intuitionistic fuzzy set, Possibility Neutrosophic set, Possibility Interval-valued Neutrosophic set, Possibility Interval-valued fuzzy set, Possibility fuzzy expert set etc., with applications in decision-making.

Data Availability

No data were used in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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