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Complex Fermatean Fuzzy Models and Their Algebraic Aggregation Operators in Decision-Making: A Case Study on COVID-19 Vaccine Selection

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Abstract: The COVID-19 pandemic has prompted extensive modeling efforts worldwide, aimed at understanding its progression and the myriad factors influencing its spread across diverse communities. The necessity for tailored control measures, varying significantly by region, became apparent early in the pandemic, leading to the implementation of diverse strategies to manage the virus both in the short and long term. The World Health Organization (WHO) has faced considerable challenges in mitigating the impact of COVID-19, necessitating adaptable and localized public health responses. Traditional mathematical models, often employing classical integer-order derivatives with real numbers, have been instrumental in analyzing the virus's spread; however, these models inadequately address the fading memory effects inherent in such complex scenarios. To overcome these limitations, fuzzy sets (FSs) were introduced, offering a robust framework for managing the uncertainty that characterizes the pandemic's dynamics. This research introduces innovative methods based on complex Fermatean FSs (CFFSs), alongside their corresponding geometric aggregation operators, including the complex Fermatean fuzzy weighted geometric aggregation (CFFWGA) operator, the complex Fermatean fuzzy ordered weighted geometric aggregation (CFFOWGA) operator, and the complex Fermatean fuzzy hybrid geometric aggregation (CFFHGA) operator. These advanced techniques are proposed as effective tools in the strategic decision-making process for reducing the spread of COVID-19. A compelling case study on COVID-19 vaccine selection was presented, demonstrating the practical applicability and superiority of these methods, effectively bridging theoretical models with real-world applications.

Keywords: CFFWGA operator, CFFOWGA operator, CFFHGA operator, Decision-making process, COVID-19 vaccine selection

1 Introduction

In December 2019, an epidemic named COVID-19 began in China and reached Pakistan by February 26, 2020, with the first case in Karachi. In March 2020, numerous cases emerged nationwide, leading to a partial lockdown to control the virus spread while maintaining essential services. The government took measures to close airspace, enforce social distancing, and improve hygiene in medical facilities. Symptoms of the virus, appearing within two to 14 days, included fever, cough, chest tightness, respiratory distress, and pneumonia. Recommended precautions were frequent hand washing, covering the mouth when coughing, self-isolation, and receiving supportive therapy.

1.1 Optimizing Emergency Response Through Advanced Modeling

COVID-19's rapid spread led to the development of various mathematical models and control techniques, including works by Toğaçar et al. [1], Ciufolini and Paolozzi [2], Tuite et al. [3], Khan and Atangana [4], Yang and Wang [5], and Sohail and Nutini [6]. Despite significant research efforts, people's understanding of COVID-19 is still limited. With ongoing clinical trials, key stages of medical technology development are crucial in advancing people's knowledge and treatment options.

1.2 Enhancing Emergency Response with Fuzzy Information Modeling

Due to limited data on COVID-19, applying existing mathematical models to analyze emergency responses and resource utilization is challenging. The WHO faces difficulties in saving lives urgently under worsening conditions. FSs are considered for their ability to manage uncertainty effectively. Experts maintain contingency plans to address issues and prevent crises, despite the challenges posed by inadequate knowledge and ambiguous information. Zadeh [7] introduced FSs for handling imprecise data but only included a membership function. Atanassov [8] improved upon traditional FSs by introducing intuitionistic FSs (IFSs), which incorporate both membership and non-membership functions. In IFSs, the total of these two functions must be less than or equal to 1. Yager [9] introduced Pythagorean FSs (PyFSs), an extension where the square of the membership and non-membership values' sum is less than or equal to 1. This modification allows for a more flexible representation of uncertainty compared to traditional FSs. PyFSs enable more nuanced decision-making by accommodating higher degrees of vagueness. Subsequent research, including work by Rahman and others, focused on applying PyFNs and mathematical models to decision-making and COVID-19 control. Several researchers developed mathematical models to understand and predict the spread of COVID-19. Rahman [10] created a model, while Ashraf and Abdullah [11] also contributed with their own model. Si et al. [12], Ashraf et al. [13] and Rahman et al. [14, 15] likewise developed different approaches to managing the pandemic's spread. These efforts collectively enhance people's ability to manage and respond to COVID-19 effectively.

1.3 Complex Fuzzy Logic for Effective Emergency Response Modeling

Fuzzy models are good at handling uncertainty and vagueness but have difficulty explicitly representing partial ignorance and how it changes over time. This challenge is particularly evident in datasets that contain periodic information. This challenge is evident in complex fields like image analysis, audio processing, and biometric databases. Ramot et al. [16] addressed this gap by introducing complex FSs (CFSs), which extend traditional FSs. CFSs offer a more advanced framework for handling uncertainty and ambiguity, providing a richer depiction of complex relationships in dynamic datasets. Alkouri and Salleh [17] introduced complex IFSs (CIFSs) by incorporating complex numbers into IFSs. This extension enhances the representation of uncertainty in decision-making by adding layers of truth and falsity. CIFs offer a more nuanced approach compared to traditional membership and non-membership degrees. Ullah et al. [18] introduced complex PyFSs (CPyFSs), which extend the concept of CIFS. CPyFSs address the limitations inherent in CIFS. This innovation enhances flexibility and applicability in various decision-making scenarios. Subsequently, Rahman et al. [19, 20] and Mahmood et al. [21] introduced various mathematical models addressing COVID-19 using CPyF information. These models provide nuanced approaches to pandemic-related data.

1.4 Study Motivation and Purpose

Building on the previous work of CPyFSs [18–21] and their application in decision-making, this study introduces CFFSs. CFFSs relax the stringent conditions of CPyFSs, providing greater flexibility and reliability in various decision-making contexts. A series of new operators were developed, which allow for more precise adjustments. As a result, CFFSs become a more powerful and effective tool for tackling decision-making problems compared to the existing sets.

1.5 Key Findings and Contributions of the Study

This study advances the field by exploring diverse aspects of CPyFSs and introducing novel techniques, enhancing the precision and applicability of fuzzy decision-making methodologies. Each contribution significantly improves the robustness and practical utility of these methods. The main contributions of the study are as follows:

i) CFFSs: CFFSs extend classical FS theory by incorporating complex numbers and Fermatean principles to handle uncertainty and ambiguity more effectively. They offer enhanced modeling capabilities for complex systems by allowing membership degrees to be expressed with greater flexibility and precision.

ii) Algebraic operational laws: These fundamental operational laws for complex Fermatean fuzzy numbers (CFFNs) are crucial for developing operators. They play a vital role in combining and processing fuzzy data effectively, enhancing the accuracy and reliability of decision-making processes.

iii) Aggregation operators: To develop a range of operators, the CFFWGA operator, CFFOWGA operator, and CFFHGA operator were introduced. These operators were designed to enhance the aggregation process in complex fuzzy environments.

iv) Algorithm for the decision-making process: An algorithm was developed based on the novel model to enhance decision-making techniques. This algorithm should leverage the improved capabilities of the model to handle complex and uncertain data, ensuring more accurate and effective decisions.

v) Visibility and reliability: To validate the visibility and reliability of the novel approach, a practical example was provided. This example illustrates how the new method effectively handles complex data and yields consistent, accurate results. By applying the approach to real-world scenarios, this study showcases its robustness and dependability, highlighting its potential for widespread application in various fields.

1.6 Structuring the Study

The rest of the study is organized as follows: Section 2 presents fundamental definitions utilized in the subsequent research. Section 3 introduces the concept of Fermatean fuzzy sets (FFSs) and outlines several of their operational laws. This section delves into the foundational principles and mathematical rules governing FFSs. It provides a thorough understanding of how these FS models function and interact. Section 4 introduces several aggregation operators within the complex Fermatean fuzzy environment, including the CFFWGA, CFFOWGA, and CFFHGA operators. These operators were designed to aggregate complex fuzzy information effectively. Section 5 presents an emergency decision-making model utilizing the novel approach introduced. This model was designed to handle urgent and critical situations more effectively. The approach aims to improve decision-making processes in high-pressure scenarios. Section 6 provides a practical example to illustrate how the concepts discussed in the preceding sections can be applied in real-world scenarios. This demonstration helps to clarify the practical utility of the theoretical ideas presented. Section 7 includes a comparative analysis and sensitivity assessment. It evaluates how different methods perform relative to each other and examines how sensitive the results are to changes in parameters. This analysis helps in understanding the robustness and reliability of the proposed approaches. Section 8 presents the conclusions of the study. It highlights the main outcomes and insights derived from the research.

2 Preliminaries

In this unit, essential definitions were laid out, which underpin the future research. These foundational terms are vital for the studies and analyses.

Definition 1 [16]: The CFS C on a universal Z can be mathematically presented as

$$C = \left\{ z, Y_C\left(z\right) e^{ia_C(z)} \middle| z \in Z \right\},$$

where $Y : Z \to [0, 1]$ is called the complex valued membership grade, which lies in a unit circle with $Y \in [0, 1]$. And $e^{ia_C(\dot{z})}$ is a real valued function.

Definition 2 [17]: The CIFS I on a universal Z can be mathematically presented as

$$I = \left\{ \left\langle z, Y_I\left(z\right) e^{ia_I(z)},\ \mathrm{T}_I\left(z\right) e^{ix_I(z)} \right\rangle \middle| z \in Z \right\},$$

where $Y, T \in [0, 1]$ are called complex valued membership grade and complex valued non-membership grade, respectively, with $a, x \in [0, 2\pi]$, $Y + \mathrm{T} \leq 1$, $\frac{a}{2\pi} + \frac{x}{2\pi} \leq 1$, and $\forall\ z \in Z$.

Definition 3 [18]: The CPyFS P on a universal Z can be mathematically presented as

$$P = \left\{ \left\langle z, P\left(z\right) e^{ia_P(z)},\ \mathrm{T}_P\left(z\right) e^{ix_P(z)} \right\rangle \middle| z \in Z \right\},$$

where $Y, T \in [0, 1]$ are called complex valued membership grade and complex valued non-membership grade, respectively, with $a, x \in [0, 2\pi]$, $Y^2 + \mathrm{T}^2 \leq 1$, $\left(\frac{a}{2\pi}\right)^2 + \left(\frac{x}{2\pi}\right)^2 \leq 1$, and $\forall$ z $\in Z$.

3 Complex Fermatean Fuzzy Model and Fundamental Operational Laws

This unit delves into the concepts of CFFS and CFFN, focusing on their operational principles. It covers the fundamental laws that govern their operations and interactions. Additionally, the unit explores how score functions and accuracy functions are employed to assess and compare these FSs and numbers, providing a framework for evaluation.

Definition 4: The CFFS F on a universal set Z can be defined as

$$F = \left\{ \left\langle z, F(z) e^{ia_F(z)}, \mathrm{T}_F(z) e^{ix_F(z)} \right\rangle \mid z \in Z \right\},$$

where $Y : Z \to [0, 1]$ and $T : Z \to [0, 1]$ present the grade of complex valued membership and complex valued non-membership of the element z in the set F with condition $Y^3 + T^3 \leq 1$. Furthermore, $a \in [0, 2\pi]$, and $x \in [0, 2\pi]$ with $\left(\frac{a}{2\pi}\right)^3 + \left(\frac{x}{2\pi}\right)^3 \leq 1$. Let $\pi = \sqrt[3]{1 - (Y^3 + T^3)} e^{\sqrt[3]{1 - \left(\left(\frac{a}{2\pi}\right)^3 + \left(\frac{x}{2\pi}\right)^3\right)}}$, then the term π is called the grade of indeterminacy or hesitancy of the element z to Z , with $\forall z \in Z$.

Figure 1 shows the comparison of proposed model with existing models.

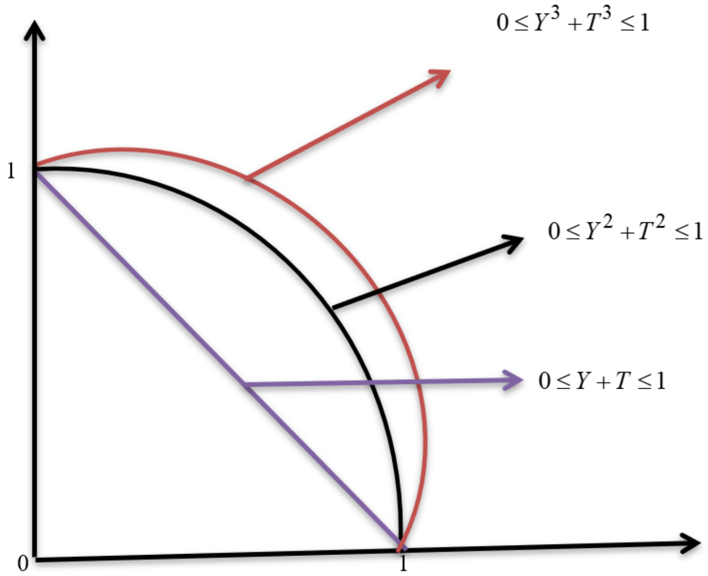

Figure 1. Space comparison of CIFS, complex Pythagorean fuzzy set (CPFS) and CFFS

Definition 5: Let $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right)$ $(j = 1, 2)$ be a family of two CFFNs and $\delta \succ 0$, then

i) $\alpha_1 \oplus \alpha_2 = \left[\sqrt[3]{Y_1^3 + Y_2^3 - Y_1^3 Y_2^3} e^{i2\pi\left(\sqrt[3]{\left(\frac{a_1}{2\pi}\right)^3 + \left(\frac{a_2}{2\pi}\right)^3 - \left(\frac{a_1}{2\pi}\right)^3\left(\frac{a_2}{2\pi}\right)^3}\right)}, (T_1 T_2)\, e^{i2\pi\left(\frac{x_1}{2\pi}\right)\left(\frac{x_2}{2\pi}\right)}\right]$

ii) $\alpha_1 \otimes \alpha_2 = \left[(Y_1 Y_2)\, e^{i2\pi\left(\frac{a_1}{2\pi}\right)\left(\frac{a_2}{2\pi}\right)}, \sqrt[3]{T_1^3 + T_2^3 - T_1^3 T_2^3} e^{i2\pi\left(\sqrt[3]{\left(\frac{x_1}{2\pi}\right)^3 + \left(\frac{x_2}{2\pi}\right)^3 - \left(\frac{x_1}{2\pi}\right)^3\left(\frac{x_2}{2\pi}\right)^3}\right)}\right]$

iii) $\delta(\alpha) = \left(\sqrt[3]{1 - (1 - Y^3)^\delta} e^{i2\pi\left(\sqrt[3]{1 - \left(1 - \left(\frac{a}{2\pi}\right)^3\right)^\delta}\right)}, (T)^\delta e^{i2\pi\left(\frac{x}{2\pi}\right)^\delta}\right)$

iv) $(\alpha)^\delta = \left((Y)^\delta e^{i2\pi\left(\frac{a}{2\pi}\right)^\delta}, \sqrt[3]{1 - (1 - T^3)^\delta} e^{i2\pi\left(\sqrt[3]{1 - \left(1 - \left(\frac{x}{2\pi}\right)^3\right)^\delta}\right)}\right)$

Definition 6: If $\alpha = \left(Ye^{ia}, Te^{ix}\right)$ is a CFFN, then the score and accuracy of α are stated as $Sc(\alpha) = \left(Y^3 - T^3\right) + \frac{1}{8\pi^3}\left(a^3 - x^3\right)$ and $Ac(\alpha) = \left(Y^3 + T^3\right) + \frac{1}{8\pi^3}\left(a^3 + x^3\right)$ respectively, satisfying the conditions, such that $Sc(\alpha) \in [-2, 2]$ and $Ac(\alpha) \in [0, 2]$.

Definition 7: If $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right)$ $(j = 1, 2)$ is a collection of CFFNs, then

i) When $Sc(\alpha_1) \succ Sc(\alpha_2) \Leftrightarrow \alpha_1 \succ \alpha_2$
ii) When $Sc(\alpha_1) \prec Sc(\alpha_2) \Leftrightarrow \alpha_1 \prec \alpha_2$
iii) When $Sc(\alpha_1) = Sc(\alpha_2)$, there are three conditions as follows:
a) $Ac(\alpha_1) \succ Ac(\alpha_2) \Leftrightarrow \alpha_1 \succ \alpha_2$
b) $Ac(\alpha_1) \prec Ac(\alpha_2) \Leftrightarrow \alpha_1 \prec \alpha_2$
c) $Ac(\alpha_1) = Ac(\alpha_2) \Leftrightarrow \alpha_1 = \alpha_2$

Property 1: Symmetry property of score function: If $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right)$ is a family of two CFFNs, and $(\alpha_j)^c = \left(T_j e^{ix_j}, Y_j e^{ia_j}\right)$ is their complement (associated inverse) function, then the following condition holds: $Sc(\alpha_1) \le Sc(\alpha_2) \Leftrightarrow Sc(\alpha_1)^c \ge Sc(\alpha_2)^c$.

Proof: By Definition 6, $Sc(\alpha_1) = \left(Y_1^3 - T_1^3\right) + \frac{1}{8\pi^3}\left(a_1^3 - x_1^3\right), Sc(\alpha_2) = \left(Y_2^3 - T_2^3\right) + \frac{1}{8\pi^3}\left(a_2^3 - x_2^3\right)$. Since $Sc(\alpha_1) \le Sc(\alpha_2)$, then the following can be deduced:

$$
\begin{aligned}
&\Leftrightarrow Sc\left(\alpha_1\right)=\left(Y_1^3-T_1^3\right)+\frac{1}{8\pi^3}\left(a_1^3-x_1^3\right)\leq Sc\left(\alpha_2\right)=\left(Y_2^3-T_2^3\right)+\frac{1}{8\pi^3}\left(a_2^3-x_2^3\right)\\
&\Leftrightarrow Sc\left(\alpha_1\right)=\left(-Y_1^3+T_1^3\right)+\frac{1}{8\pi^3}\left(-a_1^3+x_1^3\right)\geq Sc\left(\alpha_2\right)=\left(-Y_2^3+T_2^3\right)+\frac{1}{8\pi^3}\left(-a_2^3+x_2^3\right)\\
&\Leftrightarrow Sc\left(\alpha_1\right)=\left(T_1^3-Y_1^3\right)+\frac{1}{8\pi^3}\left(x_1^3-a_1^3\right)\geq Sc\left(\alpha_2\right)=\left(T_2^3-Y_2^3\right)+\frac{1}{8\pi^3}\left(x_2^3-a_2^3\right)\\
&\Leftrightarrow Sc\left(\alpha_1\right)^c\geq Sc\left(\alpha_2\right)^c
\end{aligned}
$$

Property 2: Monotonicity property of score functions: If $\alpha=\left(Ye^{ia},Te^{ix}\right)$ is a CFFN, then the score function $\mathrm{Sc}(\alpha)=\left(Y^3-T^3\right)+\frac{1}{8\pi^3}\left(a^3-x^3\right)$ monotonically decreases with T,x and monotonically increases with Y,a.

Proof: The proof is straightforward and thus is not provided.

Property 3: Symmetry property of accuracy function: If $\alpha=\left(Ye^{ia},Te^{ix}\right)$ is a CFFN, and $\alpha^c=\left(Te^{ix},Ye^{ia}\right)$ is their complement (associated inverse) function, then the following holds: $\mathrm{Ac}(\alpha)=Ac(\alpha)^c$.

Proof: By Definition 6, $Ac(\alpha)=\left(Y^3+T^3\right)+\frac{1}{8\pi^3}\left(a^3+x^3\right)$. As $Ac(\alpha)=Ac(\alpha)^c$, then $Ac(\alpha)=\left(Y^3+T^3\right)+\frac{1}{8\pi^3}\left(a^3+x^3\right)=\left(T^3+Y^3\right)+\frac{1}{8\pi^3}\left(x^3+a^3\right)=Ac(\alpha)^c$ can be obtained. Proof is completed.

Property 4: Monotonicity property of accuracy functions: If $\alpha=\left(Ye^{ia},Te^{ix}\right)$ is a CFFN. Then the accuracy function $Ac(\alpha)=\left(Y^3+T^3\right)+\frac{1}{8\pi^3}\left(a^3+x^3\right)$ monotonically increases with $Y,a,$ T and x.

Proof: The proof is straightforward and is thus not provided.

Theorem 1: Let $\alpha=\left(Ye^{ia},Te^{ix}\right)$ be a family of CFFNs, then the following can be deduced:
i) Commutative laws:
a) $\alpha_1\oplus\alpha_2=\alpha_2\oplus\alpha_1$
b) $\alpha_1\otimes\alpha_2=\alpha_2\otimes\alpha_1$
ii) Associative laws:
a) $\left(\alpha_1\oplus\alpha_2\right)\oplus\alpha_3=\alpha_1\oplus\left(\alpha_2\oplus\alpha_3\right)$
b) $\left(\alpha_1\otimes\alpha_2\right)\otimes\alpha_3=\alpha_1\otimes\left(\alpha_2\otimes\alpha_3\right)$
iii) Distributive laws:
a) $\alpha_1\otimes\left(\alpha_2\oplus\alpha_3\right)=\alpha_1\otimes\alpha_2\oplus\alpha_1\otimes\alpha_3$
b) $\left(\alpha_1\oplus\alpha_2\right)\otimes\alpha_3=\alpha_1\otimes\alpha_3\oplus\alpha_2\otimes\alpha_3$

Proof: A detailed proof for part i was provided. The methods used can be similarly applied to demonstrate parts ii and iii.
i) Since α_1 and α_2 are CFFNs, then by Definition 5 , the following can be deduced:

$$
\begin{aligned}
\alpha_1\oplus\alpha_2&=\left(\sqrt[3]{Y_1^3+Y_2^3-Y_1^3Y_2^3}e^{i2\pi\left(\sqrt[3]{\left(\frac{a_1}{2\pi}\right)^3+\left(\frac{a_2}{2\pi}\right)^3-\left(\frac{a_1}{2\pi}\right)^3\left(\frac{a_2}{2\pi}\right)^3}\right)},\left(T_1T_2\right)e^{i2\pi\left(\frac{x_1}{2\pi}\right)\left(\frac{x_2}{2\pi}\right)}\right)\\
&=\left(\sqrt[3]{Y_2^3+Y_1^3-Y_2^3Y_1^3}e^{i2\pi\left(\sqrt[3]{\left(\frac{a_2}{2\pi}\right)^3+\left(\frac{a_1}{2\pi}\right)^3-\left(\frac{a_2}{2\pi}\right)^3\left(\frac{a_1}{2\pi}\right)^3}\right)},\left(T_2T_1\right)e^{i2\pi\left(\frac{x_2}{2\pi}\right)\left(\frac{x_1}{2\pi}\right)}\right)\\
&=\alpha_2\oplus\alpha_1
\end{aligned}
$$

Theorem 2: Let $\alpha=\left(Ye^{ia},Te^{ix}\right)$ be a family of CFFNs, then $\alpha_1\otimes\alpha_2\subseteq\alpha_1\oplus\alpha_2$.

Proof: Since α_1 and α_2 are CFFNs, then by Definition 5 , the following can be deduced:

$$
\alpha_1\oplus\alpha_2=\left(\sqrt[3]{1-\prod_{j=1}^{2}\left(1-Y_j^3\right)}e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{2}1-\left(\frac{a_j}{2\pi}\right)^3}\right)},\prod_{j=1}^{2}\left(T_j\right)e^{i2\pi\prod_{j=1}^{2}\left(\frac{x_j}{2\pi}\right)}\right)
$$

$$
\alpha_1\otimes\alpha_2=\left(\prod_{j=1}^{2}\left(Y_j\right)e^{i2\pi\prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)},\sqrt[3]{1-\prod_{j=1}^{2}\left(1-T_j^3\right)}e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{2}1-\left(\frac{x_j}{2\pi}\right)^3}\right)}\right)
$$

The geometric mean is always less than or equal to the arithmetic mean for any set of positive numbers. Then $\frac{Y_1 \oplus Y_2}{2} \geq \sqrt{Y_1 Y_2}$ was obtained, this implies that $\sqrt[3]{1-\prod_{j=1}^{2}\left(1-Y_j^3\right)} \geq \prod_{j=1}^{2} Y_j$. On the similar way, it shows that the following expression holds: $\sqrt[3]{1-\prod_{j=1}^{2}\left(1-T_j^3\right)} \geq \prod_{j=1}^{2} T_j$, $2\pi\left(\sqrt[3]{1-\prod_{j=1}^{2} 1-\left(\frac{a_j}{2\pi}\right)^3}\right) \geq \prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)$ and $2\pi\left(\sqrt[3]{1-\prod_{j=1}^{2}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)}\right) \geq \prod_{j=1}^{2}\left(\frac{x_j}{2\pi}\right)$. Therefore, from the above mathematical expression, the required result was obtained, such as $\alpha_1 \otimes \alpha_2 \subseteq \alpha_1 \oplus \alpha_2$.

4 Geometric Aggregation Operators Utilizing CFFNs

This section delves into advanced aggregation operators like the CFFWGA, CFFOWGA, and CFFHGA operators, highlighting their intricate structural properties. It particularly emphasizes idempotency, a feature that guarantees consistent outcomes even when the operation is applied multiple times. Another key property is boundedness, which ensures that the results stay within a predefined range, preventing extreme values. Additionally, monotonicity was examined, a property that preserves the order of input values in the output, ensuring that the aggregated results reflect the input's relative ranking. These properties are crucial for maintaining the reliability and predictability of the aggregation process.

Definition 8: Let $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right)(1 \leq j \leq n)$ be a family of CFFNs with weighted vector $w = (w_1, w_2, \ldots, w_n)$ under conditions of $(1 \leq w_j \leq n)$ and $\sum_{j=1}^{n} w_j = 1$, then CFFWGA operator is mathematically given as follows:

$$\mathrm{CFFWGA}_w\left(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\right) = \left(\prod_{j=1}^{n}\left(Y_j\right)^{w_j} e^{i2\pi \prod_{j=1}^{n}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right)$$

Theorem 3: If a group $\alpha_j(1 \leq j \leq n)$ consists of CFFNs, then applying the CFFWGA operator to $\alpha_j(1 \leq j \leq n)$ also yields a CFFN. This ensures the result remains within the same fuzzy number framework, and

$$\mathrm{CFFWGA}_w\left(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\right) = \left(\prod_{j=1}^{n}\left(Y_j\right)^{w_j} e^{i2\pi \prod_{j=1}^{n}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right) \tag{1}$$

Proof: The principle of mathematical induction involves two main steps of proving that the base case is true and showing that it holds for the next case as well if the statement holds for one case. This establishes the truth of the statement for all natural numbers.

Step 1: For $n = 2$, if $\alpha_1 = (Y_1 e^{ia_1}, T_1 e^{ix_1})$ and $\alpha_2 = (Y_2 e^{ia_2}, T_2 e^{ix_2})$, then the following can be deduced:

$$(\alpha_1)^{w_1} = \left((Y_1)^{w_1} e^{i2\pi\left(\frac{a_1}{2\pi}\right)^{w_1}}, \sqrt[3]{1-\left(1-T_1^3\right)^{w_1}}\, e^{i2\pi\left(\sqrt[3]{1-\left(1-\left(\frac{x_1}{2\pi}\right)^3\right)^{w_1}}\right)}\right)$$

$$(\alpha_2)^{w_2} = \left((Y_2)^{w_2} e^{i2\pi\left(\frac{a_2}{2\pi}\right)^{w_2}}, \sqrt[3]{1-\left(1-T_2^3\right)^{w_2}}\, e^{i2\pi\left(\sqrt[3]{1-\left(1-\left(\frac{x_2}{2\pi}\right)^3\right)^{w_2}}\right)}\right)$$

By applying Definition 8, the following can be deduced:

$$CFFWGA_w(\alpha_1, \alpha_2) = \left(\prod_{j=1}^{2}(Y_j)^{w_j} e^{i2\pi \prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{2}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{2}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right)$$

Step 2: It holds for $n = 2$. Assuming Eq. (1) holds for $n = k$, with $k \succ 0$, then the following can be deduced:

$$\mathrm{CFFWG}A_w(\alpha_1, \alpha_2, \ldots, \alpha_k) = \left(\prod_{j=1}^{k}(Y_j)^{w_j} e^{i2\pi \prod_{j=1}^{k}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{k}\left(1-(T_j)^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right)$$

Step 3: If Eq. (1) holds for $n = k$, it shows that it is true for $n = k + 1$:

$$\begin{aligned}
&\text{CFFWGA}_w\left(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}\right) \\
&= \left(\prod_{j=1}^{k}\left(Y_j\right)^{w_j} e^{i2\pi \prod_{j=1}^{k}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{k}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{k}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right) \otimes \\
&\left(\left(Y_{k+1}\right)^{w_{k+1}} e^{i2\pi\left(\frac{a_{k+1}}{2\pi}\right)^{w_{k+1}}}, \sqrt[3]{1-\left(1-\left(T_{k+1}\right)^3\right)^{w_{k+1}}}\, e^{i2\pi\left(\sqrt[3]{1-\left(1-\left(\frac{x_{k+1}}{2\pi}\right)^3\right)^{w_{k+1}}}\right)}\right) \\
&= \left(\prod_{j=1}^{k+1}\left(Y_j\right)^{w_j} e^{i2\pi \prod_{j=1}^{k+1}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{k+1}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right)
\end{aligned}$$

Eq. (1) is true for $n = k + 1$. Thus, by the principle of mathematical induction, Eq. (1) holds for all positive integers. Thus, the proof is completed.

Then some fundamental properties were defined, including idempotency, boundedness, and monotonicity. Idempotency ensures that applying the operation multiple times yields the same result. Boundedness guarantees that the results remain within a specific range. Monotonicity ensures that the output increases with increasing input, maintaining logical consistency.

Property 1 (idempotency): If $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right)(1 \leq j \leq n)$ is a group of CFFNs, and let $\alpha_* = \left(Y_* e^{ia_*}, T_* e^{ix_*}\right)$ be another CFFN satisfying $\alpha_j = \alpha_*$, then the following can be deduced:

$$\text{CFFWGA}_w\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \left(\alpha_1\right)^{w_1} \otimes \left(\alpha_2\right)^{w_2} \otimes \ldots \otimes \left(\alpha_n\right)^{w_n} = \alpha_* \tag{2}$$

Proof: By Definition 8, the following can be deduced:

$$\begin{aligned}
&\text{CFFWGA}_w\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \\
&= \left(\prod_{j=1}^{n}\left(Y_j\right)^{w_j} e^{i2\pi \prod_{j=1}^{n}\left(\frac{a_j}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-T_j^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_j}{2\pi}\right)^3\right)^{w_j}}\right)}\right) \\
&= \left(\left(Y_*\right)^{\sum_{j=1}^{n} w_j} e^{i2\pi\left(\frac{a_*}{2\pi}\right)^{\sum_{j=1}^{n} w_j}}, \sqrt[3]{1-\left(1-T_*^3\right)^{\sum_{j=1}^{n} w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\left(1-\left(\frac{x_*}{2\pi}\right)^3\right)^{\sum_{j=1}^{n} w_j}}\right)}\right) \\
&= \left(Y_* e^{ia_*}, T_* e^{ix_*}\right) = \alpha_*
\end{aligned}$$

Property 2 (boundedness): If $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix}\right)(1 \leq j \leq n)$ is a group of some CFFNs, and let$\alpha_{\max} = \left(Y_{\max} e^{ia_{\max}}, T_{\max} e^{ix_{\max}}\right)$, $\alpha_{\min} = \left(Y_{\min} e^{ia_{\min}}, T_{\min} e^{ix_{\min}}\right)$, where $Y_{\max} = \max_j \{Y_j\}$, $a_{\max} = \max_j \{a_j\}$, $T_{\max} = \max_j \{T_j\}$, $x_{\max} = \max_j \{x_j\}$, $Y_{\min} = \min_j \{Y_j\}$, $a_{\min} = \min_j \{a_j\}$, $T_{\min} = \min_j \{T_j\}$, and $x_{\min} = \min_j \{x_j\}$, then the following can be deduced:

$$\alpha_{\min} \leq \text{CFFWGA}_w\left(w_1, w_2, w_2, \ldots, w_n\right) \leq w_{\max} \tag{3}$$

Proof: Since $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right) (1 \leq j \leq n)$, then for α_j, the following can be deduced:

$$\Leftrightarrow \sqrt[3]{\left(\min_j \{Y_{\min}\}\right)^3} \leq \sqrt[3]{Y_j^3} \leq \sqrt[3]{\left(\max_j \{Y_{\max}\}\right)^3}$$

$$\Leftrightarrow \sqrt[3]{1-\left(\max_j \{Y_{\max}\}\right)^3} \leq \sqrt[3]{1-Y_j^3} \leq \sqrt[3]{1-\left(\min_j \{Y_{\min}\}\right)^3}$$

$$\Leftrightarrow \sqrt[3]{\prod_{j=1}^{n}\left(1-\left(\max_j \{Y_{\max}\}\right)^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-Y_j^3\right)^{w_j}} \leq \sqrt[3]{\prod_{j=1}^{n}\left(1-\left(\min_j \{Y_{\min}\}\right)^3\right)^{w_j}}$$

$$\Leftrightarrow \sqrt[3]{\left(\min_j \{Y_{\min}\}\right)^3} \leq \sqrt[3]{1-\prod_{j=1}^{n}\left(1-Y_j^3\right)^{w_j}} \leq \sqrt[3]{\left(\max_j \{Y_{\max}\}\right)^3}$$

$$\min_j \{Y_{\min}\} \leq \sqrt[3]{1-\prod_{j=1}^{n}\left(1-Y_j^3\right)^{w_j}} \leq \max_j \{Y_{\max}\}$$

Thus, on the same way, it shows that $\min_j \{a_j\} \leq a_j \leq \max_j \{a_j\}$. Similarly, the following can be deduced:

$$\Leftrightarrow \min_j \{T_j\} \leq T_j \leq \max_j \{T_j\} \Leftrightarrow \prod_{j=1}^{n}\left(\min_j \{T_j\}\right)^{w_j} \leq \prod_{j=1}^{n}\left(T_j\right)^{w_j} \leq \prod_{j=1}^{n}\left(\max_j \{T_j\}\right)^{w_j}$$

$$\Leftrightarrow \left(\min_j \{T_j\}\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(T_j\right)^{w_j} \leq \left(\max_j \{T_j\}\right)^{\sum_{j=1}^{n} w_j} \Leftrightarrow \min_j \{T_j\} \leq \prod_{j=1}^{n}\left(T_j\right)^{w_j} \leq \max_j \{T_j\}$$

Thus, on the same way, $\min_j \{x_j\} \leq x_j \leq \max_j \{x_j\}$. From the above expression, $\alpha_{\min} \leq \text{CFFWG}_w\left(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\right) \leq \alpha_{\max}$ can be deduced. Thus, the proof is completed.

Property 3 (monotonicity): Suppose two different families of CFFNs, such as $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right) (1 \leq j \leq n)$ and $\alpha_j^* = \left(Y_j^* e^{ia_j^*}, T_j^* e^{ix_j^*}\right) (1 \leq j \leq n)$, satisfy the restricted conditions, such as $Y_j \leq Y_j^*$, $a_j \leq a_j^*$, $T_j^* \leq T_j$, $x_j^* \leq x_j$, then the following holds:

$$CFFWGA_w\left(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\right) \leq \text{CFFWGA}_w\left(\alpha_1^*, \alpha_2^*, \alpha_1^*, ..., \alpha_n^*\right) \tag{4}$$

Proof: Proof is similar as above. Therefore, it is omitted.

Definition 9: Let $\alpha_j = \left(Y_j e^{ia_j}, T_j e^{ix_j}\right) (1 \leq j \leq n)$ be a family of CFFNs with weighted vector $w = (w_1, w_2, ..., w_n)$ satisfying the conditions of $(1 \leq w_j \leq n)$ and $\sum_{j=1}^{n} w_j = 1$. Then CFFOWGA operator is mathematically given as follows:

$$\begin{aligned}&\text{CFFOWGA}_w\left(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\right)\\ &= \left(\prod_{j=1}^{n}\left(Y_{\sigma(j)}\right)^{w_j} e^{i2\pi \prod_{j=1}^{n}\left(\frac{a_{\sigma(j)}}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(T_{\sigma(j)}\right)^3\right)^{w_j}}\, e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_{\sigma(j)}}{2\pi}\right)^3\right)^{w_j}}\right)}\right)\end{aligned}$$

where, $(\sigma(1), \sigma(2), ..., \sigma(n))$ is any rearrangement of $(1, 2, ..., n)$ with $\alpha_{\sigma(j-1)} \geq \alpha_{\sigma(j)}$ for all j.

Theorem 4: Let $\alpha_j\ (1 \leq j \leq n)$ consist of CFFNs, then applying the CFFOWGA operator to $\alpha_j\ (1 \leq j \leq n)$ also yields a CFFN. This ensures the result remains within the same fuzzy number framework, and

CFFOWGA$_w(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n)=$

$$\left(\prod_{j=1}^{n}\left(Y_{\sigma(j)}\right)^{w_j} e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\sigma(j)}}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-T_{\sigma(j)}^3\right)^{w_j}} e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_{\sigma(j)}}{2\pi}\right)^3\right)^{w_j}}\right)}\right)$$

Proof: Refer to the proof of Theorem 3.

Property 4 (idempotency): Let $\alpha_j=\left(Y_je^{ia_j},T_je^{ix_j}\right)(1\leq j\leq n)$ be a group of CFFNs, and let $\alpha_*=\left(Y_*e^{ia_*},T_*e^{ix_*}\right)$ be another CFFN satisfying $\alpha_j=\alpha_*$, then the following can be deduced:

$$CFFOWGA_w(\alpha_1,\alpha_2,...,\alpha_n)=(\alpha_1)^{w_1}\otimes(\alpha_2)^{w_2}\otimes...\otimes(\alpha_n)^{w_n}=\alpha_* \tag{5}$$

Proof: Refer to the proof of Property 1.

Property 5 (boundedness): Let $\alpha_j=\left(Y_je^{ia_j},T_je^{ix_j}\right)(1\leq j\leq n)$ be a group of some CFFNs, and let $\alpha_{\max}=\left(Y_{\max}e^{ia_{\max}},T_{\max}e^{ix_{\max}}\right)$, $\alpha_{\min}=\left(Y_{\min}e^{ia_{\min}},T_{\min}e^{ix_{\min}}\right)$, where $Y_{\max}=\max\limits_j\{Y_j\}$, $a_{\max}=\max\limits_j\{a_j\}$, $T_{\max}=\max\limits_j\{T_j\}$, $x_{\max}=\max\limits_j\{x_j\}$, $Y_{\min}=\min\limits_j\{Y_j\}$, $a_{\min}=\min\limits_j\{a_j\}$, $T_{\min}=\min\limits_j\{T_j\}$, and $x_{\min}=\min\limits_j\{x_j\}$, then the following can be deduced:

$$\alpha_{\min}\leq \text{CFFOWGA}_w(\alpha_1,\alpha_2,\alpha_2,...,\alpha_n)\leq\alpha_{\max} \tag{6}$$

Proof: Refer to the proof of Property 2.

Property 6 (monotonicity): Let two different families of CFFNs, such as $\alpha_j=\left(Y_je^{ia_j},T_je^{ix_j}\right)(1\leq j\leq n)$ and $\alpha_j^*=\left(Y_j^*e^{ia_j^*},T_j^*e^{ix_j^*}\right)(1\leq j\leq n)$, satisfy the conditions: such as $Y_j\leq Y_j^*$, $a_j\leq a_j^*$, $T_j^*\leq T_j$, $x_j^*\leq x_j$, then the following can be deduced:

$$\text{CFFOWGA}_w(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n)\leq\text{CFFOWGA}_w(\alpha_1^*,\alpha_2^*,\alpha_3^*,...,\alpha_n^*) \tag{7}$$

Proof: Proof is easy. Therefore, it is omitted.

The aforementioned properties, such as idempotency, boundedness, and monotonicity, can be readily demonstrated for CFFHGA operators. In Definitions 8 and 9, a distinct difference emerges between the CFFWGA and CFFOWGA operators. This difference lies in the method of weight assignment: the CFFWGA operator focuses exclusively on the complex Fermatean fuzzy values themselves, whereas the CFFOWGA operator considers the weights associated with the ordered positions of these values rather than the values directly. This differentiation, however, introduces a limitation, as each set of operators concentrates on only one aspect, either the complex Fermatean fuzzy values or their ordered positions. To overcome this limitation, a novel approach is proposed, where the innovative CFFWGA and CFFOWGA operators aim to integrate and reconcile both aspects. By doing so, these operators ensure a comprehensive consideration of weights, addressing the drawback of solely focusing on either the values or their ordered positions. This integrated approach offers a more balanced and holistic treatment during the aggregation process, leading to a more nuanced and effective strategy. These operators are particularly valuable in domains requiring the aggregation of heterogeneous information from various sources, contributing to more robust and well-rounded decision-making processes. Hybrid aggregation operators, in this context, serve to combine different aggregation functions or methods, thereby enhancing performance and tackling specific challenges in data processing, decision-making, and information retrieval.

Definition 10: Let $\alpha_j=\left(Y_je^{ia_j},T_je^{ix_j}\right)(1\leq j\leq n)$ be a collection of CFFNs with weighted vector $w=(w_1,w_2,...,w_n)$ and associated vector $\nu=(\nu_1,\nu_2,...,\nu_n)$ satisfying the conditions of $(1\leq w_j\leq n)$, $\sum\limits_{j=1}^{n}w_j=1$, $(0\leq v\leq 1)$, $\sum\limits_{j=1}^{n}v_j=1$. Then CFFHGA operator is mathematically given as follows:

CFFHGA$_{v,w}(\alpha_1,\alpha_2,\alpha_3,...,\alpha_n)=$

$$\left(\prod_{j=1}^{n}\left(Y_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\dot{\alpha}_{\sigma(j)}}}{2\pi}\right)^{w_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(T_{\dot{\alpha}_{\sigma(j)}}\right)^3\right)^{w_j}} e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\frac{x_{\dot{\alpha}_{\sigma(j)}}}{2\pi}\right)^3\right)^{w_j}}\right)}\right)$$

where, $\dot{\alpha}_{\sigma(j)} = (\alpha_j)^{n\nu_j}$, and n is the balancing coefficient. If $w = (w_1, w_2, ..., w_n)$ approaches to $\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, then $((\alpha_1)^{n\nu_1}, (\alpha_2)^{n\nu_2}, ..., (\alpha_n)^{n\nu_n})$ approaches to $(\alpha_1, \alpha_2, ..., \alpha_n)$.

5 Application of the Proposed Aggregation Operators

The decision-making process involves identifying a problem or opportunity, gathering relevant information, evaluating different options, and choosing the best course of action. This systematic approach aids individuals and organizations in making well-informed and logical decisions. This study aims to enhance decision-making processes that deal with complex Fermatean fuzzy information by introducing various techniques specifically developed to handle and interpret this sophisticated data. It presents operators such as the CFFWGA, CFFOWGA, and CFFHGA, designed to address the complexities inherent in Fermatean fuzzy information. These operators offer diverse methods for decision-making, improving accuracy and adaptability in data analysis. Their specialized approaches provide more precise and flexible frameworks, enhancing decision-making processes and leading to more robust and comprehensive analyses. Ultimately, this contribution significantly advances decision-making practices.

Algorithm: Consider a set of finite alternatives $A = \{A_1, A_2, ..., A_m\}$ evaluated under specific criteria $C = \{C_1, C_2, ..., C_n\}$ with associated weights $w = (w_1, w_2, ..., w_n)$ Decision-makers in set $E = \{E_1, E_2, ..., E_k\}$ have their own weight vectors $\psi = (\psi_1, \psi_2, ..., \psi_k)^T$. The objective is to choose among the alternatives based on both the preferences of the decision-makers and the characteristics of the options. By using weighted vectors for options, criteria, and decision-makers, the process reflects the relative importance of each factor. This structured approach ensures decisions are balanced and consider all relevant aspects. Ultimately, it provides a quantifiable and systematic method for evaluating choices. To identify the best option, a multi-attribute group decision-making (MAGDM) model was created within a complex fuzzy framework (CFF) environment. This approach helps in evaluating multiple criteria with uncertainty. The decision-making process involves a series of steps that help individuals or groups make well-informed choices suited to their specific situations. This structured approach ensures a thorough evaluation of options and the selection of the best alternative. It promotes systematic and rational decision-making.

Step 1: Each decision-maker's details were organized into a matrix, where rows represent various alternatives and columns denote attributes. This layout facilitates a clear, organized comparison of options based on specific criteria.

Step 2: Individual matrices were combined into a single comprehensive matrix using the CFFWGA operator. This operator integrates data by applying weighted geometric and arithmetic means, respectively. This process improves data representation, enabling more accurate and effective decision-making.

Step 3: The CFFWGA, CFFOWGA, and CFFHGA operators were used to calculate preference values, which aid in the decision-making process. These operators aggregate complex interval-valued fuzzy data to provide precise and reliable preference scores. This ensures comprehensive integration of all relevant information for better decision outcomes.

Step 4: Using Definition 6, the score functions were calculated for all preference values to quantitatively measure how well each preference aligns with the given criteria. This assessment is essential for accurately ranking alternatives and making informed decisions. These scores guide the subsequent selection process.

Step 5: The scores of all alternatives were ranked to identify the one with the highest score. The alternative with the highest score was considered the most suitable choice. This process ensures the selection of the best option based on the evaluated criteria.

6 Illustrative Example

Case study: The problem originally tackled by Rahman et al. [15] using logarithmic operators was revisited in this study through the application of complex Fermatean fuzzy operators. This approach enables a more nuanced and flexible handling of uncertainty and imprecision, enhancing the robustness and accuracy of the decision-making process. By integrating these advanced fuzzy operators, this study aims to provide deeper insights and improved solutions compared to the traditional methods employed in previous research.

In December 2019, an epidemic began in China that would soon sweep across the globe, later named COVID-19. The term "coronavirus" was derived from the Latin word "corona" meaning "crown", "circle of light" or "nimbus" reflecting the virus's crown-like appearance under a microscope. COVID-19 did not merely disrupt daily life; it profoundly impacted the health and well-being of societies worldwide. The rapid spread of the virus led to an unprecedented global crisis, prompting the WHO to declare an international emergency. This pandemic has since presented an enormous challenge to humanity, altering the very fabric of everyday life and making the pursuit of peace and normalcy a significant struggle. As the world continues to grapple with the pandemic's effects, the need for collective action and resilience remains paramount. The disease primarily spreads through respiratory droplets when an infected person coughs, sneezes, or talks, but it can also spread via airborne transmission and contaminated surfaces. COVID-19 manifests a wide range of symptoms that vary in severity, making it a complex and challenging illness to diagnose and manage. The most common symptoms include fever, dry cough, and fatigue. However, patients may also experience a loss of taste or smell, sore throat, headache, and muscle or joint pain. Some individuals develop more severe symptoms, such as difficulty breathing, chest pain, and confusion, which require urgent medical attention.

In addition to these symptoms, COVID-19 can cause gastrointestinal issues like diarrhea and nausea, and in some cases, skin rashes or discoloration of fingers or toes. The virus affects multiple organs, leading to complications such as pneumonia, acute respiratory distress syndrome (ARDS), and multi-organ failure, particularly in individuals with underlying health conditions like diabetes, hypertension, and cardiovascular disease. Furthermore, the long-term effects of COVID-19, often referred to as "long COVID," include persistent fatigue, cognitive impairment, and respiratory problems that can last for weeks or months after the initial infection has cleared. The disease's ability to mutate has led to the emergence of several variants, some of which are more transmissible or evade immune protection, underscoring the importance of vaccination, booster doses, and public health measures to mitigate its spread. Despite global efforts, COVID-19 remains a significant public health challenge, requiring continued vigilance, research, and cooperation to manage and eventually overcome (Figure 2).

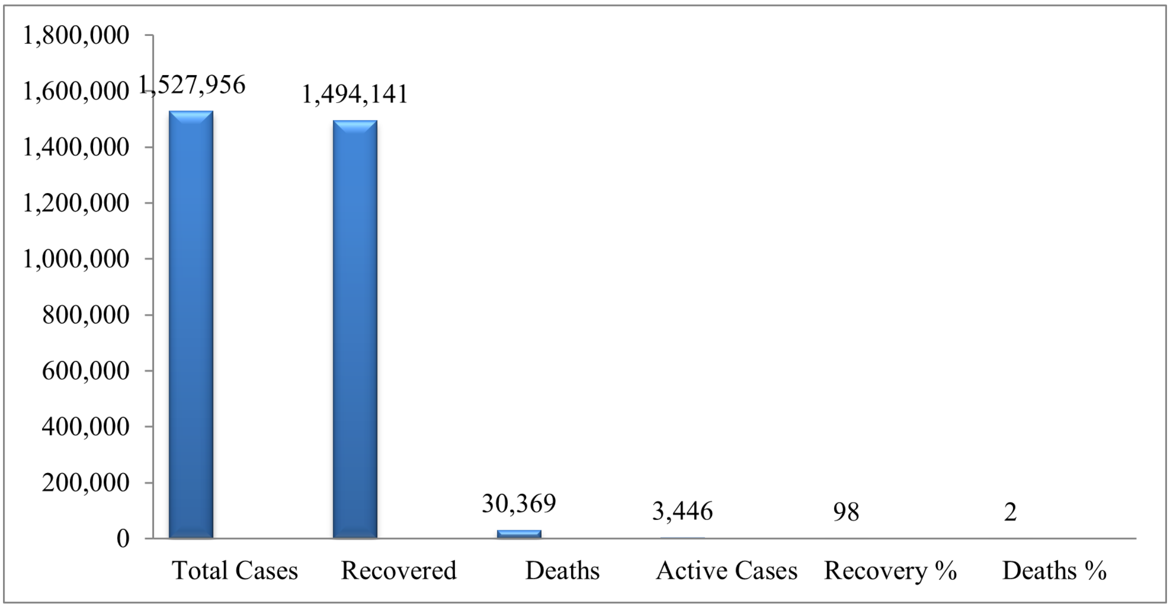

Figure 2. Covid-19 statistics in Pakistan

Efforts to control the spread of COVID-19 include vital public health measures such as social distancing, mask wearing, quarantines, and widespread vaccination campaigns. Among the vaccines developed and authorized for emergency use, several have played a critical role in reducing the transmission and severity of the virus globally. Some of the important and valuable vaccines for COVID-19 are as follows:

i) Pfizer-BioNTech Vaccine: The Pfizer-BioNTech vaccine, developed collaboratively by Pfizer and BioNTech, is an mRNA-based vaccine and among the first to receive emergency approval in multiple countries. It marked a significant milestone in COVID-19 vaccination efforts due to its rapid development and early authorization for use. This vaccine has played a crucial role in the global fight against the pandemic.

ii) Bharat Biotech's Covaxin: This vaccine, developed in India, has been utilized in numerous countries worldwide. This vaccine has played a significant role in the global fight against COVID-19. Its development and distribution highlight India's contribution to international public health efforts.

iii) AstraZeneca (Vaxzevria): AstraZeneca's Vaxzevria, developed in partnership with the University of Oxford, is a viral vector vaccine that has seen widespread use globally. This collaboration produced a COVID-19 vaccine that has been distributed in numerous countries, contributing significantly to vaccination efforts. The vaccine employs a modified virus to stimulate an immune response, offering protection against COVID-19.

iv) Sinopharm: The Sinopharm COVID-19 vaccine, industrialized by China's National Pharmaceutical Group (Sinopharm), is an inactivated virus vaccine. Utilizing a killed virus to trigger an immune response without causing illness, this vaccine is a key component in the global fight against COVID-19. Produced and distributed by the Chinese government, the Sinopharm vaccine contributes significantly to worldwide efforts to control the pandemic.

v) Sinovac (CoronaVac): CoronaVac, developed by Sinovac Biotech, is an inactivated virus vaccine aimed at combating COVID-19. It employs a killed form of the coronavirus to trigger an immune response. This method is among the various strategies used to offer protection against the virus. By using a non-live version of the pathogen, it helps the immune system recognize and fight the actual virus if encountered. This approach contributes to the diverse arsenal of vaccines available for COVID-19.

vi) Moderna Vaccine: The Moderna vaccine, another mRNA-based vaccine, was granted emergency use authorization to help combat the COVID-19 pandemic. It functions by directing cells to produce a specific protein that stimulates an immune response. This immune reaction equips the body to recognize and combat the virus more effectively if it is encountered in the future.

vii) Johnson & Johnson (Janssen): The Johnson & Johnson vaccine, developed by its subsidiary Janssen Pharmaceuticals, is administered as a single-dose shot. This vaccine offers the convenience of requiring only one dose, simplifying the vaccination process. Janssen's development provides a crucial option in the global effort to combat the pandemic.

Similar to other countries, Pakistan aims to curb the spread of COVID-19 by selecting the most effective vaccine. To achieve this, the government formed a committee of four expert doctors $\{E_1, E_2, E_3, E_4\}$ with respective weights of $\psi = (0.1, 0.2, 0.3, 0.4)$. This committee was tasked with evaluating and selecting the most suitable vaccine for the country from the available options. In the initial selection, the experts identified four valuable and more suitable vaccines: A_1: Pfizer BioNTech Vaccine, A_2: Astra Zeneca vaccine, $[A_3$: Sinovac (CoronaVac), A_4: Moderna Vaccine. These choices reflect the expert consensus on the most effective and reliable options available. Selecting vaccines for a population involves evaluating several critical criteria, including safety, efficacy, and practical aspects of distribution and administration. Experts made decisions on the four shortlisted vaccines based on five specific criteria, each with an assigned weight vector $w = (0.1, 0.2, 0.2, 0.2, 0.3)$. These criteria ensure a comprehensive assessment of each vaccine's overall suitability. The weighted vector helps prioritize factors based on their importance, guiding experts in making well-informed choices. This approach ensures that the selected vaccines are not only effective and safe but also practical for widespread use.

i) Age group (C_1): Certain vaccines are tailored to specific age groups, with some being designed for infants and others for adults. This ensures that the vaccine's efficacy and safety are optimized for the particular age group it targets. Thus, it's crucial to consider the intended age group when selecting vaccines.

ii) International collaboration (C_2): International collaboration with other countries and organizations can enhance access to a broader spectrum of vaccines and resources. Such partnerships can streamline efforts, share knowledge, and optimize resource utilization. This collective approach can significantly improve the effectiveness of vaccination programs globally.

iii) Safety (C_3): Ensuring the safety of a vaccine is crucial. It undergoes extensive testing in clinical trials to confirm it does not cause significant adverse effects. A comprehensive and well-documented safety profile is essential for any vaccine.

iv) Availability of the vaccine (C_4): The availability of the vaccine can differ depending on the region and the time of year. Factors such as local supply chains, distribution logistics, and regional demand can all influence when and where the vaccine is accessible. Consequently, individuals may experience varying levels of access to C_4 based on their specific location and timing.

v) Efficacy (C_5): The vaccine's effectiveness in preventing the targeted disease is a crucial consideration. High efficacy indicates that the vaccine offers substantial protection against both infection and severe illness. This strong protective ability is essential for controlling the spread and impact of the disease.

Step 1: This involves creating a detailed compilation of information about all the experts, which is presented in Tables 1- 4. These tables were designed to organize and display essential details about each expert in a structured matrix format.

Step 2: In this step, Table 5 was obtained using the CFFWGA operator with weights $\psi = (0.1, 0.2, 0.3, 0.4)$. This process involves applying the specified weights to various elements. The resulting table displays the weighted aggregation of these elements.

Table 1. Assessment of E_1

	C_1	C_2	C_3	C_4	C_5
A_1	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.68e^{i2\pi(0.53)}, \\ 0.58e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.53e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0.53)}, \\ 0.72e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.58e^{i2\pi(0.63)}, \\ 0.57e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.64e^{i2\pi(0.53)}, \\ 0.53e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.63)}, \\ 0.48e^{i2\pi(0.49)} \end{pmatrix}$

Table 2. Assessment of E_2

	C_1	C_2	C_3	C_4	C_5
A_1	$\begin{pmatrix} 0.64e^{i2\pi(0.53)}, \\ 0.53e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.63)}, \\ 0.48e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.58e^{i2\pi(0.63)}, \\ 0.57e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.53e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0.53)}, \\ 0.72e^{i2\pi(0.46)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.68e^{i2\pi(0.53)}, \\ 0.58e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$

Table 3. Assessment of E_3

	C_1	C_2	C_3	C_4	C_5
A_1	$\begin{pmatrix} 0.65e^{i2\pi(0.45)}, \\ 0.51e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.63e^{i2\pi(0.45)}, \\ 0.50e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.68e^{i2\pi(0.53)}, \\ 0.58e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.58e^{i2\pi(0.63)}, \\ 0.57e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.72e^{i2\pi(0.53)}, \\ 0.60e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.56e^{i2\pi(0.63)}, \\ 0.67e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0.53)}, \\ 0.72e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.64e^{i2\pi(0.53)}, \\ 0.53e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$

Table 4. Assessment of E_4

	C_1	C_2	C_3	C_4	C_5
A_1	$\begin{pmatrix} 0.58e^{i2\pi(0.63)}, \\ 0.57e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.63)}, \\ 0.48e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.64e^{i2\pi(0.53)}, \\ 0.53e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.67)}, \\ 0.61e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0.53)}, \\ 0.72e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$	$\begin{pmatrix} 0.58e^{i2\pi(0.63)}, \\ 0.57e^{i2\pi(0.49)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.63)}, \\ 0.56e^{i2\pi(0.49)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$	$\begin{pmatrix} 0.60e^{i2\pi(0.53)}, \\ 0.72e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.63e^{i2\pi(0.67)}, \\ 0.58e^{i2\pi(0.50)} \end{pmatrix}$	$\begin{pmatrix} 0.64e^{i2\pi(0.53)}, \\ 0.53e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.49e^{i2\pi(0.45)}, \\ 0.63e^{i2\pi(0.51)} \end{pmatrix}$

Table 5. Collective assessment of all experts

	C_1	C_2	C_3	C_4	C_5
A_1	$\begin{pmatrix} 0.77e^{i2\pi(0.77)} \\ 0.59e^{i2\pi(0.73)} \end{pmatrix}$	$\begin{pmatrix} 0.74e^{i2\pi(0.59)} \\ 0.76e^{i2\pi(0.70)} \end{pmatrix}$	$\begin{pmatrix} 0.75e^{i2\pi(0.57)} \\ 0.56e^{i2\pi(0.76)} \end{pmatrix}$	$\begin{pmatrix} 0.76e^{i2\pi(0.74)} \\ 0.71e^{i2\pi(0.68)} \end{pmatrix}$	$\begin{pmatrix} 0.66e^{i2\pi(0.70)} \\ 0.72e^{i2\pi(0.75)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.63e^{i2\pi(0.58)} \\ 0.70e^{i2\pi(0.79)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.77)} \\ 0.73e^{i2\pi(0.70)} \end{pmatrix}$	$\begin{pmatrix} 0.82e^{i2\pi(0.66)} \\ 0.66e^{i2\pi(0.73)} \end{pmatrix}$	$\begin{pmatrix} 0.62e^{i2\pi(0.69)} \\ 0.80e^{i2\pi(0.68)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.59)} \\ 0.77e^{i2\pi(0.86)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.66e^{i2\pi(0.65)} \\ 0.75e^{i2\pi(0.83)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0.60)} \\ 0.70e^{i2\pi(0.86)} \end{pmatrix}$	$\begin{pmatrix} 0.83e^{i2\pi(0.55)} \\ 0.61e^{i2\pi(0.81)} \end{pmatrix}$	$\begin{pmatrix} 0.83e^{i2\pi(0.80)} \\ 0.51e^{i2\pi(0.77)} \end{pmatrix}$	$\begin{pmatrix} 0.71e^{i2\pi(0.80)} \\ 0.60e^{i2\pi(0.70)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.73e^{i2\pi(0.61)} \\ 0.75e^{i2\pi(0.84)} \end{pmatrix}$	$\begin{pmatrix} 0.75e^{i2\pi(0.67)} \\ 0.77e^{i2\pi(0.88)} \end{pmatrix}$	$\begin{pmatrix} 0.75e^{i2\pi(0.42)} \\ 0.70e^{i2\pi(0.85)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.77)} \\ 0.71e^{i2\pi(0.70)} \end{pmatrix}$	$\begin{pmatrix} 0.67e^{i2\pi(0.71)} \\ 0.64e^{i2\pi(0.81)} \end{pmatrix}$

Step 3: To use the CFFWGA operator with weights $w = (0.1, 0.2, 0.2, 0.2, 0.3)$, the preferences need to be aggregated based on their cumulative fuzzy weights. The preference values were determined by the degree of satisfaction each element contributes in the fuzzy aggregation process.

$$f_1 = \left(0.72e^{i2\pi(0.76)}, 0.69e^{i2\pi(0.64)}\right), f_2 = \left(0.76e^{i2\pi(0.68)}, 0.61e^{i2\pi(0.57)}\right)$$

$$f_3 = \left(0.76e^{i2\pi(0.78)}, 0.63e^{i2\pi(0.59)}\right), f_4 = \left(0.71e^{i2\pi(0.68)}, 0.60e^{i2\pi(0.62)}\right)$$

Step 4: This step involves using Definition 6 to evaluate the alignment of each preference value with the specified criteria. Each preference value was assigned a score based on its adherence to the criteria detailed in Definition 5. These scores reflect how well each preference value meets the defined standards.

$$scor\,(f_1) = \left((0.72)^3 - (0.69)^3\right) + \frac{1}{8\pi^3}\left((0.76)^3 - (0.64)^3\right) = 0.22$$
$$scor\,(f_2) = \left((0.76)^3 - (0.61)^3\right) + \frac{1}{8\pi^3}\left((0.68)^3 - (0.57)^3\right) = 0.34$$
$$scor\,(f_3) = \left((0.76)^3 - (0.63)^3\right) + \frac{1}{8\pi^3}\left((0.78)^3 - (0.59)^3\right) = 0.45$$
$$scor\,(f_4) = \left((0.71)^3 - (0.60)^3\right) + \frac{1}{8\pi^3}\left((0.68)^3 - (0.62)^3\right) = 0.21$$

Step 5: Thus, the best option for COVID-19 vaccination in Pakistan is Sinovac (CoronaVac), given its proven efficacy and availability. This vaccine has been widely administered, showing positive results in controlling the spread of the virus.

Table 6 presents an overview of the score functions used by different methods, showcasing their effectiveness in evaluating and ranking various alternatives. This comparative analysis offers a clear understanding of the performance of each method, aiding the decision-making process by pinpointing the most suitable option. This helps in selecting the best alternative based on a thorough evaluation of each method's capabilities (Table 6).

Table 6. Covid-19 statistical analysis of Pakistan

Province	Total Cases	Recoveries	Active Cases	Deaths	Recovery	Deaths
Punjab	506,018	491,193	1,265	13,560	97 %	3 %
Sindh	576,769	566,956	1,712	8,101	98 %	1 %
KPK	219,460	212,756	380	6,324	97 %	3 %
Baluchistan	35,484	35,105	1	378	99 %	1 %
Islamabad	135,178	134,089	66	1,023	99 %	1 %
AJK	43,310	42,510	8	792	98 %	2 %
GB	11,737	11,532	14	191	98 %	2 %

7 Comparative and Sensitivity Analyses

A CFFS is an advanced extension of various FS theories, providing a more detailed and versatile framework. Traditional FSs are limited to representing membership degrees, which indicate how much an element belongs to a set. However, more sophisticated FS models have been developed to include additional information, leading to richer representations. For instance, IFS and PFS incorporate both membership and non-membership degrees, offering a clearer picture by quantifying both the degree of belonging and the degree of non-belonging of elements. Similarly, FFS also uses both types of degrees, allowing for even more nuanced decision-making. CFS takes a different approach by utilizing complex numbers to express membership degrees. This allows for the inclusion of phase information, which can represent additional aspects of uncertainty or preference. Building on this, CIFS and CPFS extend the concept further by incorporating complex-valued membership and non-membership degrees, subject to certain constraints. In CIFS, the sum of the absolute squares of these degrees must be less than or equal to one, while in CPFS, the sum of the squares themselves must be constrained. The CFFS model introduces another layer of sophistication by using complex-valued membership and non-membership degrees with the sum of their cubes constrained to be less than or equal to one. This unique constraint allows CFFS to capture and represent more detailed two-dimensional information, making it a more generalized and flexible model compared to its predecessors. By employing these complex-valued degrees and specific constraints, CFFS can handle a broader range of scenarios and uncertainties, providing a powerful tool for various applications in decision-making, pattern recognition, and more. This enhancement over traditional FSs and their subsequent extensions positions CFFS as a comprehensive and versatile framework within FS theory, capable of addressing more complex and nuanced problems effectively. Therefore, the proposed model demonstrates greater flexibility and adaptability compared to existing models, as illustrated in Table 7 and Figure 3. This enhanced versatility makes it more effective in various applications. Table 8 shows the performance evaluation of various operators. Table 9 shows the comparative and sensitivity analyses.

Table 7. Score functions of all methods

	CFFWGA	CFFOWGA	CFFHGA
A_1	0.22	0.25	0.27
A_2	0.34	0.32	0.36
A_3	0.45	0.39	0.41
A_4	0.21	0.26	0.29

Table 8. Performance evaluation of various operators

Operators	Score Functions	Ranking
CFFWGA	$\text{scor}\,(f_3) \succ \text{scor}\,(f_2) \succ \text{scor}\,(f_1) \succ \text{scor}\,(f_4)$	$A_3 \succ A_2 \succ A_1 \succ A_4$
CFFOWGA	$\text{scor}\,(f_3) \succ \text{scor}\,(f_2) \succ \text{scor}\,(f_1) \succ \text{scor}\,(f_4)$	$A_3 \succ A_2 \succ A_1 \succ A_4$
CFFHGA	$\text{scor}\,(f_3) \succ \text{scor}\,(f_2) \succ \text{scor}\,(f_1) \succ \text{scor}\,(f_4)$	$A_3 \succ A_2 \succ A_1 \succ A_4$

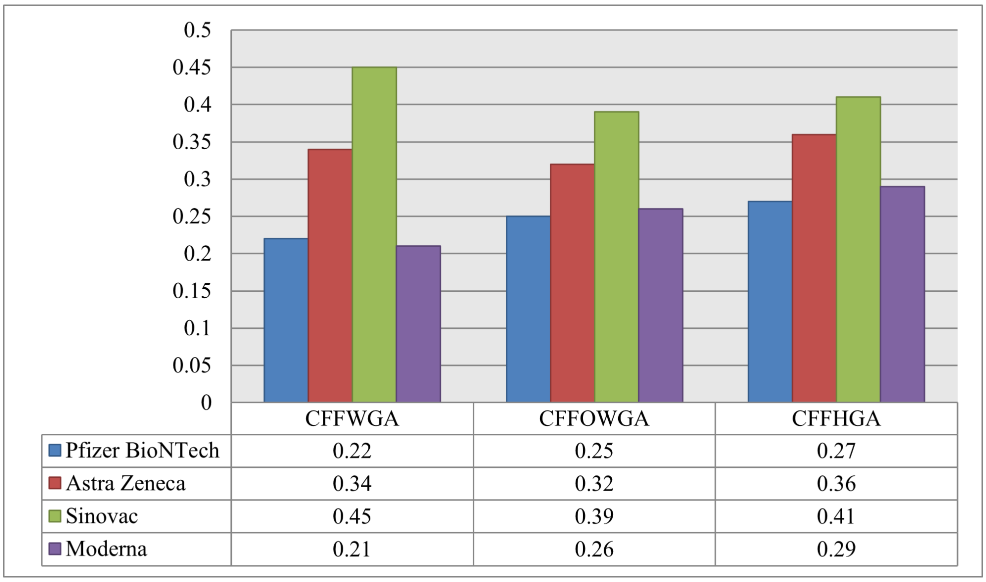

Figure 3. Ranking of all methods

Table 9. Comparative and sensitivity analyses

Sets	Uncertainty	Falsity	Indeterminacy	Periodicity	2-D Information	Power in Cube
FSs	✓	×	×	×	×	×
IFSs	✓	✓	✓	×	×	×
ByESs	✓	✓	✓	×	×	×
FFSs	✓	✓	✓	×	×	×
CFSs	✓	×	×	✓	✓	×
CIFSs	✓	✓	✓	✓	✓	×
CPyESS	✓	✓	✓	✓	✓	×
CFFSs	✓	✓	✓	✓	✓	✓

8 Conclusion

In March 2020, several COVID-19 cases were identified in hospitals across Khyber Pakhtunkhwa, Pakistan, prompting urgent action from the Pakistani government. In response, authorities implemented strict measures to curb the virus's spread nationwide, including lockdowns, travel restrictions, and public awareness campaigns, aiming to mitigate the pandemic's impact and protect the nation's health infrastructure. Measures include implementing strict health protocols, enhancing testing and tracing efforts, and promoting public awareness campaigns. The government also considered lockdowns and travel restrictions to contain the outbreak. These comprehensive steps are crucial to controlling the pandemic and protecting public health. To address the need for an effective COVID-19 vaccine, the Pakistani government decided to identify the most suitable option for patients. To achieve this, complex Fermatean fuzzy information was employed based on algebraic techniques. In this innovative study, a fuzzy mathematical model was developed within a complex fuzzy environment, utilizing algebraic methods. Several new algebraic techniques were introduced based on CFFNs, including the CFFWGA operator, the CFFOWGA operator, and the CFFHGA operator. These operators facilitate a more precise evaluation and selection process, ensuring the chosen vaccine is effective and reliable for COVID-19 patients. At the conclusion of this study, a COVID-19 emergency position description was presented to demonstrate the practicality and reliability of the results. This description underscores the effectiveness and efficiency of the unique approaches developed. By applying the methods to real-world scenarios, this study highlights their value and applicability in addressing urgent and complex challenges. This reinforces the robustness and utility of the proposed models, showcasing their potential to provide actionable solutions in critical situations.

In the future, it is planned to apply complex Fermatean fuzzy models to a variety of practical applications, expanding their current utilization, including real-world scenarios such as medical diagnosis, pattern recognition, machine learning, and brain hemorrhage detection. By adapting these techniques to environments that utilize fuzzy logic extensions, the research aims to improve their versatility and performance. This approach will allow for more accurate and reliable decision-making across diverse fields. Ultimately, the goal is to enhance the effectiveness of complex Fermatean fuzzy models in addressing complex problems in various settings.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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