Knowledge SuperHypergraphs, Multimodal SuperHypergraphs, Lattice SuperHypergraphs, and Hyperbolic SuperHypergraphs: Concepts and Applications

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Abstract: This work builds on hypergraphs—graphs whose edges can link any number of vertices—and superhypergraphs, which add a recursive, hierarchical powerset structure to hyperedges. It reviews four practical hypergraph variants: Knowledge Hypergraphs (for multi-relational knowledge representation), Multimodal Hypergraphs (for combining different data modalities), Lattice Hypergraphs (for spatial and topological modeling), and Hyperbolic Hypergraphs (for embedding vertices in hyperbolic space to capture hierarchies). The paper then shows how to elevate each of these into the superhypergraph framework—resulting in Knowledge SuperHypergraphs, Multimodal SuperHypergraphs, Lattice SuperHypergraphs, and Hyperbolic SuperHypergraphs—and outlines their core properties. Overall, it offers a unified, more expressive modeling approach that paves the way for future advances in both hypergraph and superhypergraph research.

Keywords: Superhypergraph; Hypergraph; Graph class; Knowledge graphs; Multimodal graphs; Lattice graphs; Hyperbolic graphs

1 Introduction

1.1 Graph Theory and Hypergraph Theory

Graph theory remains a vibrant area of research. Graphs can model relationships between objects, leading to a wide range of practical applications. A hypergraph generalizes the notion of a graph by allowing edges—called hyperedges—to connect any number of vertices rather than just two [1–5].

Numerous real-world applications of hypergraphs and well-suited graph classes have been extensively studied. A Knowledge Hypergraph extends traditional knowledge graphs by encoding higher-order relationships among entities using hyperedges, enabling expressive multi-relational knowledge representation in structured datasets [6, 7]. A Multimodal Hypergraph integrates multiple hypergraphs—each representing a different modality—by assigning weights to their hyperedges, facilitating comprehensive modeling of heterogeneous, multi-relational data [8, 9]. A Lattice Hypergraph represents discrete spatial units and their topological relationships, combining local geometric adjacency with higher-order connectivity via hyperedges for structured modeling [10–12]. A Hyperbolic Hypergraph embeds vertices in hyperbolic space, leveraging its negative curvature to capture hierarchical and complex relational patterns [13, 14].

A SuperHyperGraph is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model [15–18]. Like hypergraphs, superhypergraphs have also attracted active research attention [19–21].

1.2 Our Contributions

Research on hypergraphs and superhypergraphs is crucial both for modeling real-world hierarchical concepts and for exploring the mathematical richness of their structures. However, studies on SuperHyperGraphs remain limited. In particular, the superhypergraph analogues of Knowledge Hypergraphs, Multimodal Hypergraphs, Lattice Hypergraphs, and Hyperbolic Hypergraphs have not yet been investigated.

In this paper, we extend Knowledge Hypergraphs, Multimodal Hypergraphs, Lattice Hypergraphs, and Hyperbolic Hypergraphs using the SuperHyperGraph framework, introducing Knowledge SuperHypergraphs, Multimodal SuperHypergraphs, Lattice SuperHypergraphs, and Hyperbolic SuperHypergraphs. We provide a concise examination of their properties and demonstrate how these new classes can serve as a foundation for future research on both Hypergraphs and SuperHypergraphs.

2 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, we deal only with finite structures. Unless otherwise specified, n is assumed to be a non-negative integer. For fundamental operations, concepts, and principles of graphs, refer to Diestel [22].

2.1 Graphs and Hypergraphs

In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between elements, thereby enhancing its utility in various fields [1, 4]. In the following, we present rigorous definitions for graphs, subgraphs, and hypergraphs. In this paper, we focus on finite, undirected, and simple graphs.

Definition1 Graph [22]. A graph G is defined as an ordered pair

$$G = (V, E), \tag{1}$$

where, V is a set of vertices and E is a set of edges. Each edge $e \in E$ connects two distinct vertices from V.

Definition2 Subgraph [22]. Given a graph $G = (V, E)$, a subgraph $H = (V_H, E_H)$ of G satisfies:

• $V_H \subseteq V$; that is, the vertices of H form a subset of those of G.

• $E_H \subseteq E$; the edges of H are taken from those of G.

• Every edge in E_H has both endpoints contained in V_H.

Furthermore, H is called induced if for any two vertices $u, v \in V_H$, the edge (u, v) belongs to E_H exactly when it is an edge in G.

Definition3 Hypergraph [1, 23]. A hypergraph $H = (V(H), E(H))$ consists of:

• A nonempty set $V(H)$ of vertices.

• A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

2.2 Powerset and n-th Powerset

In what follows, we employ the concepts of the powerset and the n-th powerset as fundamental building blocks for our later constructions.

Definition4 Set [24]. A set is a collection of distinct objects, called elements, which are unambiguously defined. If A is a set and x is an element of A, we write $x \in A$. Sets are usually denoted by enclosing their elements in curly braces.

Definition5 Subset [24]. For any two sets A and B, A is said to be a subset of B (written $A \subseteq B$) if every element of A is also an element of B:

$$\forall x \in A, \quad x \in B. \tag{2}$$

If additionally $A \neq B$, then A is called a proper subset of B, denoted $A \subset B$.

Definition6 Empty Set [24]. The empty set, denoted $\emptyset$, is the unique set that contains no elements:

$$\forall x, \quad x \notin \emptyset. \tag{3}$$

It follows that $\emptyset$ is a subset of every set.

Definition7 Base Set. A base set S is the underlying set from which more elaborate structures, such as powersets and hyperstructures, are constructed. It is defined by

$$S = \{x \mid x \text{ belongs to a specified domain}\}. \tag{4}$$

All elements appearing in constructions like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are drawn from S.

Definition8 Powerset. The powerset of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including both $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}. \tag{5}$$

Definition9 n-th Powerset [25–28]. The n-th powerset of a set H, denoted $P_n(H)$, is defined recursively by:

$$P_1(H) = \mathcal{P}(H), \quad P_{n+1}(H) = \mathcal{P}(P_n(H)) \quad \text{for } n \geq 1. \tag{6}$$

Similarly, the n-th nonempty powerset, denoted $P_n^*(H)$, is given by:

$$P_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(P_n^*(H)), \tag{7}$$

where, $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set omitted.

2.3 Hyperstructure and Superhyperstructure

To develop a comprehensive framework for hyperstructures [29–31] and superhyperstructures [25, 32], we now introduce several key definitions. In this context, a hypergraph may be viewed as a hyperstructure, while a superhypergraph is naturally regarded as a superhyperstructure.

Definition10 Classical Structure [25, 33]. A classical structure is a mathematical system based on a nonempty set H that is endowed with one or more classical operations satisfying a prescribed set of axioms. A classical operation is a mapping:

$$\#_0 : H^m \to H, \tag{8}$$

where, $m \geq 1$ and H^m denotes the m-fold Cartesian product of H. Typical examples include operations like addition or multiplication in algebraic systems such as groups, rings, or fields.

Definition11 Hyperoperation [34].

A hyperoperation is a generalization of a binary operation in which the combination of two elements yields a set of elements rather than a single element. Formally, for a set S, a hyperoperation $\circ$ is defined by:

$$\circ : S \times S \to \mathcal{P}(S), \tag{9}$$

where, $\mathcal{P}(S)$ represents the powerset of S.

Definition12 Hyperstructure [25, 35].

A hyperstructure is an extension of a classical structure where the operations are defined on the powerset of a base set. It is formally given by:

$$\mathcal{H} = (\mathcal{P}(S), \circ), \tag{10}$$

with S as the base set and $\circ$ acting on subsets of S.

Definition13 SuperHyperOperations [25]. Let H be a nonempty set and $P(H)$ its powerset. Define the n-th powerset $P^n(H)$ recursively by:

$$P^0(H) = H, \quad P^{k+1}(H) = P(P^k(H)) \quad \text{for } k \geq 0. \tag{11}$$

A SuperHyperOperation of order (m, n) is an m-ary operation.

$$o^{(m,n)} : H^m \to P_*^n(H), \tag{12}$$

where, $P_*^n(H)$ denotes the n-th powerset of H (either excluding the empty set, which we refer to as a classical-type operation, or including it, known as a Neutrosophic-type operation). These operations serve as higher-order generalizations of hyperoperations by capturing multi-level complexity through iterative powerset constructions.

Definition14 n-Superhyperstructure [25, 33]. An n-superhyperstructure generalizes a hyperstructure by incorporating the n-th powerset of a base set. It is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ), \tag{13}$$

where, S is the base set, $\mathcal{P}_n(S)$ is its n-th powerset, and $\circ$ is an operation on elements of $\mathcal{P}_n(S)$.

2.4 SuperHyperGraph

A SuperHyperGraph is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model [36, 37]. This concept has been recently introduced and extensively studied in the literature [36, 38–41].

Definition15 n-SuperHyperGraph [38].

Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\left(\mathcal{P}^k(V_0)\right), \tag{14}$$

where, $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An n-SuperHyperGraph is then a pair.

$$\text{SHT}^{(n)} = (V, E), \tag{15}$$

with,

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0). \tag{16}$$

Each element of V is called an n-supervertex and each element of E an n-superedge.

Theorem1. Let V_0 be a finite base set and let an n-SuperHyperGraph be defined as

$$\text{SHT}^{(n)} = (V, E), \tag{17}$$

with,

$$V, E \subseteq \mathcal{P}^n(V_0), \tag{18}$$

where, $\mathcal{P}^n(V_0)$ denotes the nth powerset of V_0. Then there exists a transformation (via a flattening function F) that maps $\text{SHT}^{(n)}$ into a classical hypergraph.

$$H = (V_0, E'), \tag{19}$$

where,

$$E' = \{F(e) \mid e \in E\} \quad \text{and} \quad F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0) \tag{20}$$

is defined recursively by

$$F(x) = x \quad \text{for } x \in V_0, \quad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0) \text{ with } k \geq 1. \tag{21}$$

Proof. We prove the theorem by constructing the flattening function F and verifying that it transforms each n-superedge into a subset of the base set V_0.

Step 1: Definition of the Flattening Function. For any element $x \in V_0$ (i.e. a base element), define

$$F(x) = x. \tag{22}$$

Now, for any $A \in \mathcal{P}^k(V_0)$ with $k \geq 1$ (i.e. an element of the kth powerset), define recursively

$$F(A) = \bigcup_{a \in A} F(a). \tag{23}$$

In particular, for an n-supervertex or n-superedge $e \in \mathcal{P}^n(V_0)$, the function $F(e)$ returns a subset of V_0.

Step 2: Transformation of the n-SuperHyperGraph. Given the n-SuperHyperGraph $\text{SHT}^{(n)} = (V, E)$ with $V, E \subseteq \mathcal{P}^n(V_0)$, define the new set of hyperedges by

$$E' = \{F(e) \mid e \in E\} \tag{24}$$

Since each $e \in E$ is an element of $\mathcal{P}^n(V_0)$, by applying F we obtain $F(e) \subseteq V_0$. Therefore, $E' \subseteq \mathcal{P}(V_0)$.

Step 3: Forming the Classical Hypergraph. Define the classical hypergraph

$$H = (V_0, E'), \tag{25}$$

where, the vertex set is V_0 (the original base set) and each hyperedge in E' is a subset of V_0. By the definition of a hypergraph, H is a valid hypergraph.

The flattening function F thus transforms every n-superedge $e \in E$ into an ordinary hyperedge $F(e)$ on V_0. Hence, every n-SuperHyperGraph $\text{SHT}^{(n)}$ is transformed into a classical hypergraph $H = (V_0, E')$.

2.5 Knowledge Hypergraph

A knowledge hypergraph extends knowledge graphs by encoding higher-order relationships among entities using hyperedges, enabling expressive multi-relational knowledge representation in structured datasets [6, 7, 42]. A related concept is the knowledge graph [43–45], which is well-known in the literature. The definition is given as follows.

Definition16 Knowledge Hypergraph. Let
- E be a finite set of **entities**,
- R be a finite set of **relations**.

For each relation $r \in R$, let $|r| \in \mathbb{N}$ denote its **arity**, i.e., the number of arguments (entities) that participate in the relation.

Define the set of all possible tuples (facts) as

$$\tau = \left\{ r\left(e_1, e_2, \ldots, e_{|r|}\right) \mid r \in R, e_i \in E \text{ for } i = 1, \ldots, |r| \right\}. \tag{26}$$

A **world** is a complete assignment of truth values to every tuple in τ; that is, for each $x \in \tau$, the tuple x is either true or false.

A **knowledge hypergraph** is then defined as the triple.

$$\mathcal{H} = (E, R, \tau_0). \tag{27}$$

where, $\tau_0 \subseteq \tau$ is the set of tuples that are true in the world. In other words, each true fact $r(e_1, e_2, \ldots, e_{|r|}) \in \tau_0$ represents a relation among the entities $e_1, e_2, \ldots, e_{|r|}$. Note that when every relation has arity 2, i.e. $|r| = 2$ for all $r \in R$, the knowledge hypergraph coincides with the standard knowledge graph.

Example1 Knowledge Hypergraph for Industrial Collaboration. Consider a simplified scenario in which a technology industry analyst wants to capture complex relationships among people, companies, projects, and locations. We model this using a knowledge hypergraph $\mathcal{H} = (E, R, \tau_0)$ as follows:

Entities.

$$E = \{\text{Alice, Bob, Carol, InnoTechInc, FutureAI Ltd, ProjectX, ProjectY, NewYork, SanFrancisco}\}. \tag{28}$$

here,
- Alice, Bob, Carol are researchers or engineers.
- InnoTechInc, FutureAI Ltd are companies.
- ProjectX, ProjectY are collaborative research projects.
- NewYork, SanFrancisco are cities where companies are headquartered.

Relations.

$$R = \{\text{worksAt, headquarteredIn, collaboratesOn, launchedBy}\}. \tag{29}$$

Each relation $r \in R$ has the following arity $|r|$:
- $|\text{worksAt}| = 2$: connects a person to a company.
- $|\text{headquarteredIn}| = 2$: connects a company to a city.
- $|\text{collaboratesOn}| = 3$: connects two people and a project.
- $|\text{launchedBy}| = 3$: connects a project, a company, and a year.

All Possible Tuples.

$$\tau = \left\{ r\left(e_1, e_2, \ldots, e_{|r|}\right) \mid r \in R, e_i \in E \right\}. \tag{30}$$

For instance:

$$\text{worksAt}(\text{Alice, InnoTechInc}),$$
$$\text{headquarteredIn}(\text{FutureAI Ltd, SanFrancisco}),$$
$$\text{collaboratesOn}(\text{Alice, Bob, ProjectX}),$$
$$\text{launchedBy}(\text{ProjectY, InnoTechInc, 2023}), \quad \text{etc.}$$

True Facts. We define $\tau_0 \subseteq \tau$ to be the set of true tuples (hyperedges) in the current world. For example:

$$\tau_0 = \{ \text{worksAt}(\text{Alice, InnoTechInc}), \text{worksAt}(\text{Bob, FutureAI Ltd}), \text{worksAt}(\text{Carol, FutureAI Ltd}),$$
$$\text{headquarteredIn}(\text{InnoTechInc, NewYork}), \text{headquarteredIn}(\text{FutureAI Ltd, SanFrancisco}),$$
$$\text{collaboratesOn}(\text{Alice, Bob, ProjectX}), \text{collaboratesOn}(\text{Bob, Carol, ProjectY}),$$
$$\text{launchedBy}(\text{ProjectX, InnoTechInc, 2022}), \text{launchedBy}(\text{ProjectY, FutureAI Ltd, 2023}) \}.$$

Each hyperedge in τ_0 encodes a higher-order relationship:
- worksAt(Alice, InnoTechInc) indicates that Alice is employed by InnoTechInc.
- headquarteredIn(FutureAI Ltd, SanFrancisco) encodes the location of FutureAI Ltd.
- collaboratesOn(Alice, Bob, ProjectX) means Alice and Bob collaborate on ProjectX together.
- launchedBy(ProjectY, FutureAI Ltd, 2023) represents that ProjectY was launched by FutureAI Ltd in 2023.

Interpretation. This knowledge hypergraph captures not only pairwise relations (e.g., worksAt, headquarteredIn) but also ternary relations (collaboratesOn and launchedBy) that require three entities. For instance, the hyperedge

$$\text{collaboratesOn(Bob, Carol, ProjectY)}$$

encodes that Bob and Carol jointly collaborate on ProjectY. By allowing arity $|r| > 2$, the hypergraph naturally represents complex, real-world facts without the need for intermediate nodes or reification.

Comparison to a Knowledge Graph. If every relation had arity exactly 2 (i.e., $|r| = 2$ for all $r \in R$), then $\mathcal{H}$ reduces to a standard knowledge graph. In our example, the binary facts worksAt(p, c) and headquaredIn(c, City) would correspond to ordinary edges. However, modeling collaboratesOn or launchedBy as binary edges in a knowledge graph would require additional intermediary nodes or reified statements, complicating both storage and querying. The hypergraph approach keeps each multi-entity relation as a single hyperedge, preserving expressiveness and clarity.

2.6 Multimodal Hypergraph

A multimodal hypergraph integrates multiple hypergraphs, each representing a different modality, by assigning weights to their hyperedges. This approach enables comprehensive multi-relational and heterogeneous data modeling [8, 9]. Related concepts, such as multimodal graphs [46, 47], are well-known in the literature.

Definition 17 Multimodal Hypergraph [48]. Let V be a finite set of vertices. For each modality $m = 1, 2, \ldots, M$, let

$$G_m = (V, E_m, W_m) \,. \tag{31}$$

be a hypergraph defined on V, where:
- E_m is a set of hyperedges, and each hyperedge $e \in E_m$ is a non-empty subset of V (typically, $|e| \geq 2$);
- $W_m : E_m \to \mathbb{R}^+$ is a function assigning a positive weight to each hyperedge $e \in E_m$.

Furthermore, let $\{\alpha_m\}_{m=1}^M$ be a set of combination weights such that

$$\alpha_m \geq 0 \quad \text{for all } m = 1, \ldots, M, \quad \text{and} \quad \sum_{m=1}^M \alpha_m = 1. \tag{32}$$

The multimodal hypergraph is then defined as the tuple

$$\mathcal{G} = \left(V, \{E_m\}_{m=1}^M, \{W_m\}_{m=1}^M, \{\alpha_m\}_{m=1}^M \right) \tag{33}$$

which integrates the individual modality-specific hypergraphs by weighting each G_m according to α_m. In certain applications, a unified representation is obtained via a combined Laplacian matrix given by

$$\mathcal{L} = \sum_{m=1}^M \alpha_m \mathcal{L}_m, \tag{34}$$

where, $\mathcal{L}_m$ is the Laplacian matrix corresponding to the hypergraph G_m.

Example 2 Multimodal Hypergraph for Social Media Analysis. Consider a social media platform where we wish to model interactions among users, posts, and topics through multiple modalities: user–post engagement, post–topic tagging, and user–user friendships. We build a multimodal hypergraph $\mathcal{G} = (V, \{E_m\}_{m=1}^3, \{W_m\}_{m=1}^3, \{\alpha_m\}_{m=1}^3)$ as follows:

Vertices.

$$V = \{u_1, u_2, u_3, u_4, p_1, p_2, p_3, p_4, t_1, t_2, t_3\} \,, \tag{35}$$

where, each u_i is a user, each p_j is a post, and each t_k is a topic.

Modality 1: User–Post Engagement ($m = 1$).
- Hyperedges E_1 group users who have liked or commented on the same post:

$$E_1 = \{e_{1,1} = \{u_1, u_2, p_1\}, e_{1,2} = \{u_2, u_3, p_2\}, e_{1,3} = \{u_1, u_3, u_4, p_3\}, e_{1,4} = \{u_4, p_4\}\} \,. \tag{36}$$

- Weights $W_1(e)$ reflect the total number of interactions on that post:

$$W_1(e_{1,1}) = 5, \quad W_1(e_{1,2}) = 3, \quad W_1(e_{1,3}) = 7, \quad W_1(e_{1,4}) = 2. \tag{37}$$

Modality 2: Post–Topic Tagging ($m = 2$).
- Hyperedges E_2 group posts that share the same topic:

$$E_2 = \{e_{2,1} = \{p_1, p_2, t_1\}, e_{2,2} = \{p_2, p_3, t_2\}, e_{2,3} = \{p_3, p_4, t_3\}\} \tag{38}$$

- Weights $W_2(e)$ reflect the relevance score of the topic to those posts (e.g., TF-IDF average):

$$W_2(e_{2,1}) = 0.8, \quad W_2(e_{2,2}) = 0.6, \quad W_2(e_{2,3}) = 0.9. \tag{39}$$

Modality 3: User–User Friendship ($m = 3$).
- Hyperedges E_3 group small user communities (triads or pairs) of friends:

$$E_3 = \{e_{3,1} = \{u_1, u_2, u_3\}, e_{3,2} = \{u_2, u_4\}, e_{3,3} = \{u_3, u_4\}\} \tag{40}$$

- Weights $W_3(e)$ denote the strength of friendship (e.g., frequency of direct messages):

$$W_3(e_{3,1}) = 10, \quad W_3(e_{3,2}) = 4, \quad W_3(e_{3,3}) = 6. \tag{41}$$

Combination Weights. Choose combination weights $\{\alpha_m\}_{m=1}^3$ to reflect the relative importance of each modality. For instance:

$$\alpha_1 = 0.5, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.2, \quad \sum_{m=1}^3 \alpha_m = 1. \tag{42}$$

Combined Laplacian (Optional). If one wishes to perform spectral clustering over this multimodal hypergraph, the combined Laplacian matrix is

$$\mathcal{L} = \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3, \tag{43}$$

where, each $\mathcal{L}_m$ is the Laplacian of the hypergraph $G_m = (V, E_m, W_m)$.

Interpretation.
- The first modality ($m = 1$) captures groups of users interacting on the same post p_j. For example, $e_{1,1} = \{u_1, u_2, p_1\}$ indicates that users u_1 and u_2 both engaged with post p_1.
- The second modality ($m = 2$) captures topic clusters: $e_{2,1} = \{p_1, p_2, t_1\}$ means that posts p_1 and p_2 are both tagged with topic t_1.
- The third modality ($m = 3$) captures friendship relations: $e_{3,1} = \{u_1, u_2, u_3\}$ indicates a tight-knit triad of friend connections among users u_1, u_2, u_3.
- By weighting each modality with α_m, one can control, for example, whether user–post engagement ($\alpha_1 = 0.5$) or topic tagging ($\alpha_2 = 0.3$) should have more influence in downstream tasks such as recommendation or community detection.

2.7 Lattice Hypergraph

A lattice is a discrete mathematical structure where points are arranged in a regular, repeating pattern, often used in algebra, geometry, and physics [49, 50]. A Lattice Hypergraph represents discrete spatial units and their topological relationships, integrating local geometric adjacency and higher-order connectivity via hyperedges for structured modeling [10–12].

Definition18 Lattice Hypergraph. Let

$$V_c = \{v_1, v_2, \ldots, v_{N_c}\} \tag{44}$$

be a finite set of lattice vertices, where each vertex v_i represents a discrete spatial unit (or grid cell) in a geometric layout. In applications such as VLSI design, each lattice vertex corresponds to a G-cell—an area with fixed spatial coordinates in a regular grid arrangement.

Let

$$V_n = \{u_1, u_2, \ldots, u_{N_n}\} \tag{45}$$

be a finite set of hyperedge nodes, where each u_j represents a grouping of lattice vertices that share a common topological or connectivity property (for example, a G-net that covers multiple G-cells connected via a netlist).

Define the lattice adjacency matrix $A \in \{0,1\}^{N_c \times N_c}$ by

$$A_{ij} = \begin{cases} 1, & \text{if lattice vertices } v_i \text{ and } v_j \text{ are geometrically adjacent (e.g., share a common boundary),} \\ 0, & \text{otherwise.} \end{cases} \quad (46)$$

This matrix captures the local, spatial connectivity between neighboring lattice vertices.

Define the incidence matrix $H \in \{0,1\}^{N_c \times N_n}$ by

$$H_{ij} = \begin{cases} 1, & \text{if lattice vertex } v_i \text{ is a member of the hyperedge associated with } u_j \\ 0, & \text{otherwise} \end{cases} \quad (47)$$

The incidence matrix H encodes the higher-order topological relationships by linking lattice vertices to hyperedge nodes.

Then, the lattice hypergraph is defined as the heterogeneous graph

$$\mathcal{G} = (V_c, V_n, A, H), \quad (48)$$

which jointly models:

• **Local geometric interactions** via the lattice adjacency matrix A, and

• **Higher-order topological interactions** via the incidence matrix H.

This formulation enables information (or message passing) to propagate both among spatially adjacent regions and across groups of regions that are topologically connected by shared hyperedges.

Example3 Lattice Hypergraph for VLSI Routing.

Consider a simplified VLSI (Very Large Scale Integration) chip layout, where the chip surface is divided into a regular grid of G-cells (grid cells). We wish to model both the local adjacency of those cells and the higher-order connectivity imposed by routing nets. We construct a lattice hypergraph $\mathcal{G} = (V_c, V_n, A, H)$ as follows:

Lattice Vertices (V_c).

Partition the chip area into a 3×3 grid of G-cells:

$$V_c = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\} \quad (49)$$

We index them row by row:

$v_1 = $ cell at (row 1, column 1), $\quad v_2 = $ cell at (row 1, column 2), $\quad v_3 = $ cell at (row 1, column 3),

$v_4 = $ cell at (row 2, column 1), $\quad v_5 = $ cell at (row 2, column 2), $\quad v_6 = $ cell at (row 2, column 3),

$v_7 = $ cell at (row 3, column 1), $\quad v_8 = $ cell at (row 3, column 2), $\quad v_9 = $ cell at (row 3, column 3).

Hyperedge Nodes (V_n).

Suppose there are two routing nets on this chip:

$$V_n = \{u_1, u_2\}, \quad (50)$$

where,

• u_1 (Net 1) connects G-cells $\{v_2, v_5, v_8\}$ in a vertical net,

• u_2 (Net 2) connects G-cells $\{v_4, v_5, v_6\}$ in a horizontal net.

Lattice Adjacency Matrix A.

Each G-cell is adjacent (shares a boundary) with its immediate orthogonal neighbors. For our 3×3 grid:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \quad (51)$$

For example, $A_{1,2} = 1$ since v_1 (row 1, col 1) is adjacent to v_2 (row 1, col 2); $A_{5,8} = 1$ since v_5 (row 2, col 2) is adjacent to v_8 (row 3, col 2), and so on.

Incidence Matrix H.

We encode membership of G-cells in each net:

$$H = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \\ H_{3,1} & H_{3,2} \\ H_{4,1} & H_{4,2} \\ H_{5,1} & H_{5,2} \\ H_{6,1} & H_{6,2} \\ H_{7,1} & H_{7,2} \\ H_{8,1} & H_{8,2} \\ H_{9,1} & H_{9,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{52}$$

Row i corresponds to v_i, and column j to u_j. For instance:

$$H_{2,1} = 1 \quad \text{since } v_2 \in u_1; \quad H_{5,2} = 1 \quad \text{since } v_5 \in u_2; \quad H_{5,1} = 1 \quad \text{since } v_5 \in u_1. \tag{53}$$

Interpretation.

• The adjacency matrix A models local geometric interactions among neighboring G-cells. For example, v_5 (center cell) is adjacent to v_2, v_4, v_6, v_8.

• The incidence matrix H captures higher-order topological connectivity: $\{v_2, v_5, v_8\}$ form a hyperedge u_1, reflecting Net 1's vertical connection; $\{v_4, v_5, v_6\}$ form a hyperedge u_2, reflecting Net 2's horizontal connection.

• By combining A and H, one can propagate signals or perform optimization that respects both local adjacency (e.g. heat dissipation or local routing congestion) and net-level grouping (e.g. ensure all cells in a net are connected).

Thus, the lattice hypergraph $\mathcal{G} = (V_c, V_n, A, H)$ provides a unified representation of the VLSI layout, integrating both spatial adjacency and netlist connectivity in a single heterogeneous framework.

2.8 Hyperbolic Hypergraph

A Hyperbolic Hypergraph is a hypergraph in which vertices are embedded in hyperbolic space, utilizing its negative curvature to facilitate hierarchical and complex relational modeling [13, 14, 51]. Related concepts, such as Hyperbolic Graphs [52–54], are well-known in the literature.

Definition19 Hyperbolic Hypergraph [13]. Let V be a finite set of vertices and let E be a collection of hyperedges such that each hyperedge

$$e \in E \text{ is a subset of } V \text{ with } |e| \geq 2. \tag{54}$$

Let $\mathbb{H}^d$ denote a d-dimensional hyperbolic space; that is, a complete Riemannian manifold with constant negative curvature $-\kappa$ (with $\kappa > 0$). A hyperbolic hypergraph is defined as the triple

$$\mathcal{H}_{\mathbb{H}} = (V, E, \phi), \tag{55}$$

where, $\phi : V \to \mathbb{H}^d$ is an injective mapping (or embedding) that assigns to each vertex $v \in V$ a unique point $\phi(v)$ in $\mathbb{H}^d$. The hyperbolic distance between any two vertices $u, v \in V$ is given by

$$d_{\mathrm{H}}(\phi(u), \phi(v)). \tag{56}$$

Moreover, for each hyperedge $e \in E$, one may define an optional hyperedge representation $\mu(e)$ (for example, the hyperbolic Fréchet mean of the set $\{\phi(v) : v \in e\}$) to capture the collective geometry of the vertices in e. This structure leverages the rich geometry of hyperbolic space to model complex, higher-order, and often hierarchical relationships among vertices, which is especially useful in applications such as sequential recommendation.

Example4 Hyperbolic Hypergraph for Hierarchical Product Categories. Consider an online retailer's inventory organized into a hierarchical product taxonomy. We wish to model both the hierarchical relationships among product categories and the co-occurrence of items within categories. We construct a hyperbolic hypergraph $\mathcal{H}_{\mathbb{H}} = (V, E, \phi)$ as follows:

Vertices. Let

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \tag{57}$$

where, each v_i represents a specific product. For instance:

- $v_1 = \text{Smartphone}_A$,
- $v_2 = \text{Smartphone}_B$,
- $v_3 = \text{Laptop}_A$,
- $v_4 = \text{Laptop}_B$,
- $v_5 = \text{Headphones}_A$,
- $v_6 = \text{Headphones}_B$,
- $v_7 = \text{Tablet}_A$.

Hyperedges. Define hyperedges E that reflect category groupings:

$$E = \{e_1, e_2, e_3, e_4\}, \tag{58}$$

where,
- $e_1 = \{v_1, v_2\}$ groups all "Smartphones,"
- $e_2 = \{v_3, v_4\}$ groups all "Laptops,"
- $e_3 = \{v_5, v_6\}$ groups all "Headphones,"
- $e_4 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ represents the top-level "Electronics" category.

Embedding ϕ. Embed each vertex v_i into the 2D Poincaré disk model of hyperbolic space $\mathbb{H}^2$. For concreteness, choose coordinates that reflect the taxonomy's hierarchy:

$$\phi(v_1) = (r = 0.9,\ \theta = 0°), \qquad \phi(v_2) = (r = 0.9,\ \theta = 20°), \qquad \phi(v_3) = (r = 0.9,\ \theta = 160°),$$

$$\phi(v_4) = (r = 0.9,\ \theta = 180°), \quad \phi(v_5) = (r = 0.9,\ \theta = 200°),$$

$$\phi(v_6) = (r = 0.9,\ \theta = 340°), \quad \phi(v_7) = (r = 0.5,\ \theta = 0°).$$

Here (r, θ) denotes radial and angular coordinates in the Poincaré disk:

$$\phi(v) = x + iy, \quad x = \frac{r \cos\theta}{1 + \sqrt{1 - r^2}}, \quad y = \frac{r \sin\theta}{1 + \sqrt{1 - r^2}} \tag{59}$$

Hyperbolic Distances. The hyperbolic distance between any two embedded vertices $u, v \in V$ is given by

$$d_{\mathbb{H}}(\phi(u), \phi(v)) = \text{arcosh}\left(1 + 2 \frac{\|\phi(u) - \phi(v)\|^2}{(1 - \|\phi(u)\|^2)(1 - \|\phi(v)\|^2)}\right). \tag{60}$$

For example, computing

$$\phi(v) = x + iy, \quad x = \frac{r \cos\theta}{1 + \sqrt{1 - r^2}}, \quad y = \frac{r \sin\theta}{1 + \sqrt{1 - r^2}} \tag{61}$$

reveals that $\|\phi(v_1) - \phi(v_2)\|$ is small in hyperbolic terms (both near the "Smartphone" boundary), whereas v_7 (Tablet) is more centrally placed and thus farther from v_1.

Hyperedge Representations. For each hyperedge $e \in E$, define a representative point $\mu(e) \in \mathbb{H}^2$ as the hyperbolic Fréchet mean of its member vertices:

$$\mu(e) = \arg\min_{z \in \mathbb{H}^2} \sum_{v \in e} d_{\mathbb{H}}(z, \phi(v))^2. \tag{62}$$

Concretely:
- For $e_1 = \{v_1, v_2\}$, the Fréchet mean $\mu(e_1)$ lies slightly inward along the geodesic between $\phi(v_1)$ and $\phi(v_2)$.
- For $e_4 = \{v_1, \ldots, v_7\}$, the Fréchet mean $\mu(e_4)$ is near the center of the disk (small r), reflecting the top-level "Electronics" category.

Interpretation.
- Vertices v_1 and v_2 (two smartphone models) are embedded close to the boundary of $\mathbb{H}^2$ at similar angles, capturing their fine-grained similarity within the "Smartphone" subcategory.
- Vertex v_7 (Tablet_A) is placed more centrally (smaller radial coordinate), indicating it belongs to a higher-level category ("Electronics") but is not tightly nested under "Smartphone" or "Laptop".
- Hyperedge e_1 (all smartphones) corresponds to a small arc near the boundary; its Fréchet mean $\mu(e_1)$ is slightly interior but still relatively large r.
- Hyperedge e_4 (top-level category "Electronics") is very "broad" and its Fréchet mean $\mu(e_4)$ is near the origin, reflecting that "Electronics" spans all subcategories.
- The negative curvature of $\mathbb{H}^2$ naturally places more specific categories (smartphones, laptops, headphones) near the boundary and more general categories nearer the center, capturing hierarchical multi-scale relationships.

Thus, the hyperbolic hypergraph $\mathcal{H}_{\mathbb{H}} = (V, E, \phi)$ models both higher-order groupings (hyperedges) and a natural hierarchy via hyperbolic embedding. This is especially useful for recommendation tasks, where distances in $\mathbb{H}^2$ correlate with product similarity at multiple granularities.

3 Results

In this paper, we introduce several new graph classes by extending the hypergraphs defined in the previous section to superhypergraphs. Throughout this paper, we consider only finite structures. Unless otherwise noted, n is assumed to be a non-negative integer.

3.1 Knowledge n-SuperHyperGraph

The definition of the Knowledge n-SuperHyperGraph is given as follows.

Definition20 Knowledge n-SuperHyperGraph. Let E_0 be a finite set of base entities and R_0 be a finite set of base relations, where each relation $r \in R_0$ has an associated arity $|r| \in \mathbb{N}$. Define the set of all facts (or tuples) in the usual manner by

$$\tau = \left\{ r\left(e_1, e_2, \ldots, e_{|r|}\right) \mid r \in R_0, e_i \in E_0 \text{ for } i = 1, \ldots, |r| \right\} \tag{63}$$

For an integer $n \geq 0$, define the n-th powerset of E_0 recursively by

$$\mathcal{P}^0\left(E_0\right) = E_0, \quad \mathcal{P}^{n+1}\left(E_0\right) = \mathcal{P}\left(\mathcal{P}^n\left(E_0\right)\right), \tag{64}$$

and similarly for R_0.

A knowledge n-SuperHyperGraph is the structure

$$\mathcal{KH}^{(n)} = \left(V, R, \tau_0^{(n)}\right), \tag{65}$$

where,
- $V \subseteq \mathcal{P}^n(E_0)$ is the set of n-supervertices (which represent entities at a higher-order level),
- $R \subseteq \mathcal{P}^n(R_0)$ is the set of n-superrelations (with arity defined over elements of V),
- $\tau_0^{(n)} \subseteq \left\{ r(v_1, v_2, \ldots, v_{|r|}) \mid r \in R, \ v_i \in V \right\}$ is the set of true n-superfacts.

Example5 Knowledge 1-SuperHyperGraph. Let the base set of entities be

$$E_0 = \{a, b\}, \tag{66}$$

and let the base set of relations be

$$R_0 = \{r\}, \tag{67}$$

where, r is a binary relation. Then the first powerset of E_0 is

$$\mathcal{P}^1\left(E_0\right) = \mathcal{P}\left(E_0\right) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \tag{68}$$

Define the set of 1-supervertices by excluding the empty set:

$$V = \{\{a\}, \{b\}, \{a, b\}\}. \tag{69}$$

For simplicity, let the set of 1-superrelations be

$$R = \{\{r\}\}. \tag{70}$$

Finally, define the set of true 1-superfacts as

$$\tau_0^{(1)} = \{r(\{a\}, \{b\})\}. \tag{71}$$

Then the structure

$$\mathcal{KH}^{(1)} = \left(V, R, \tau_0^{(1)}\right) \tag{72}$$

is a concrete example of a Knowledge 1-SuperHyperGraph.

Example6 Knowledge 2-SuperHyperGraph for Department-Level Collaborations. We model an academic-industry collaboration scenario in which individual researchers are grouped into research teams, and those teams are further organized into departments. A knowledge 2-SuperHyperGraph captures "department-level" relations (e.g., collaboration) by nesting sets twice.

Base Entities E_0. Let

$$E_0 = \{\text{Alice, Bob, Carol, Dave, Eve}\} \tag{73}$$

be the set of individual researchers.

Base Relation R_0. We have a single binary relation

$$R_0 = \{ \text{ collaborates } \}, \tag{74}$$

where, $|\text{collaborates}| = 2$. A fact $\text{collaborates}(x, y)$ means "researcher x collaborates with researcher y."

First Powerset $\mathcal{P}^1(E_0)$.

$$\mathcal{P}^1(E_0) = \mathcal{P}(E_0) \tag{75}$$

includes, for example, $\{ \text{Alice}, \text{Bob}\}, \{\text{Carol}\}, \{\text{Dave}, \text{Eve}\}$, etc. We interpret certain subsets as "research teams".

Let:

$$T_1 = \{ \text{ Alice }, \text{ Bob } \}, \quad T_2 = \{ \text{ Carol } \}, \quad T_3 = \{ \text{ Dave }, \text{ Eve } \} \tag{76}$$

Second Powerset $\mathcal{P}^2(E_0)$.

$$\mathcal{P}^2(E_0) = \mathcal{P}(\mathcal{P}(E_0)) \tag{77}$$

whose elements are sets of subsets of E_0. Two relevant elements of $\mathcal{P}^2(E_0)$ are:

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\}, \tag{78}$$

where, $T_1, T_2, T_3 \in \mathcal{P}^1(E_0)$. We interpret D_1 and D_2 as two "departments," each grouping two teams.

Set of 2-Supervertices V. Choose

$$V = \{D_1, D_2\}, \quad D_1 = \{\{ \text{ Alice }, \text{ Bob } \}, \{ \text{ Carol } \}, \quad D_2 = \{\{ \text{ Carol } \}, \{ \text{ Dave }, \text{ Eve } \}\} \tag{79}$$

Each $D_i \subseteq \mathcal{P}^1(E_0)$ lies in $\mathcal{P}^2(E_0)$. We call these the 2-supervertices (departments).

Set of 2-Superrelations R. Form the set of second-level relations by taking a singleton subset of R_0 inside $\mathcal{P}^1(R_0)$, then again inside $\mathcal{P}^2(R_0)$. Concretely:

$$R = \{\{\{ \text{ collaborates } \}\}\} \subseteq \mathcal{P}^2(R_0) \tag{80}$$

Denote $s = \{\{\text{collaborates}\}\}$. Its arity—as an n-superrelation—remains 2, now interpreted on pairs of 2-supervertices.

Flattening Function F. Recall $F : \mathcal{P}^2(E_0) \to \mathcal{P}(E_0)$ is defined by

$$F(x) = \bigcup_{a \in x} F(a), \quad F(a) = a \text{ for } a \in \mathcal{P}^1(E_0) \tag{81}$$

Thus:

$$
\begin{aligned}
F(D_1) &= F(\{\{\text{Alice}, \text{Bob}\}, \{\text{Carol}\}\}) = \{\text{Alice}, \text{Bob}\} \cup \{\text{Carol}\} = \{\text{Alice}, \text{Bob}, \text{Carol}\}, \\
F(D_2) &= \{\text{Carol}\} \cup \{\text{Dave}, \text{Eve}\} = \{\text{Carol}, \text{Dave}, \text{Eve}\}.
\end{aligned}
\tag{82}
$$

Set of True 2-SuperFacts $\tau_0^{(2)}$. We choose one true superfact indicating that department D_1 collaborates with department D_2. Since $s = \{\{\text{collaborates}\}\}$, we write:

$$\tau_0^{(2)} = \{s(D_1, D_2)\} \tag{83}$$

The triple:

$$\mathcal{KH}^{(2)} = \left(V, R, \tau_0^{(2)}\right) \tag{84}$$

is a concrete knowledge 2-SuperHyperGraph.

Interpretation.

• D_1 represents "Department 1," which consists of Team 1 $\{ \text{Alice}, \text{Bob}\}$ and Team 2 $\{\text{Carol}\}$. After flattening, $F(D_1) = \{\text{Alice}, \text{Bob}, \text{Carol}\}$.

• D_2 represents "Department 2," which consists of Team 2 $\{\text{Carol}\}$ and Team 3 $\{\text{Dave}, \text{Eve}\}$. After flattening, $F(D_2) = \{\text{Carol}, \text{Dave}, \text{Eve}\}$.

• $s = \{\{\text{collaborates}\}\}$ is the unique 2-superrelation indicating "collaboration" at the department level.

• The superfact $s(D_1, D_2)$ in $\tau_0^{(2)}$ means "Department 1 collaborates with Department 2." Concretely, it implies that every pair of researchers (x, y) with $x \in F(D_1)$, $y \in F(D_2)$, has an underlying base-level relation `collaborates`(x, y).

Thus, $\mathcal{KH}^{(2)} = (V, R, \tau_0^{(2)})$ provides a two-level nesting of individuals into teams and teams into departments, with a department-level collaboration relation capturing higher-order group interactions in a single, unified structure.

Theorem2. Every knowledge HyperGraph $\mathcal{H} = (E, R, \tau_0)$ is a special case of a knowledge n-SuperHyperGraph. In particular:

1. For $n = 0$, we have $\mathcal{P}^0(E_0) = E_0$ and $\mathcal{P}^0(R_0) = R_0$; hence,

$$\mathcal{KH}^{(0)} = (E, R, \tau_0), \tag{85}$$

which is exactly the classical knowledge HyperGraph.

2. For any $n \geq 1$, there exists a canonical embedding

$$i : E_0 \rightarrow \mathcal{P}^n(E_0) \tag{86}$$

defined by iteratively taking singletons (i.e., $i(e) = \{ \{ \cdots \{e\} \cdots \} \}$ with n iterations). This embedding is injective and allows us to identify each entity $e \in E_0$ with an n-supervertex $i(e) \in \mathcal{P}^n(E_0)$. By embedding relations and facts in a similar fashion, the original knowledge HyperGraph $\mathcal{H}$ can be seen as isomorphic to a substructure of the knowledge n-SuperHyperGraph.

$$\mathcal{KH}^{(n)} = \left(V, R', \tau_0^{(n)} \right), \tag{87}$$

where,

$$V = i(E_0) \subseteq \mathcal{P}^n(E_0) \quad \text{and} \quad \tau_0^{(n)} = \left\{ r\left(i(e_1), \ldots, i\left(e_{|r|}\right)\right) \mid r\left(e_1, \ldots, e_{|r|}\right) \in \tau_0 \right\}. \tag{88}$$

Thus, the classical knowledge HyperGraph is naturally generalized by the knowledge n-SuperHyperGraph construction.

Proof. Case 1: $n = 0$. By definition, $\mathcal{P}^0(E_0) = E_0$ and $\mathcal{P}^0(R_0) = R_0$. Therefore, the knowledge 0-SuperHyperGraph is

$$\mathcal{KH}^{(0)} = (E_0, R_0, \tau_0) \tag{89}$$

which is exactly the original knowledge HyperGraph $\mathcal{H}$.

Case 2: $n \geq 1$. Define the canonical embedding

$$i : E_0 \rightarrow \mathcal{P}^n(E_0) \tag{90}$$

by setting, for each $e \in E_0$,

$$i(e) = \{\{\cdots \{e\} \cdots\}\}, \tag{91}$$

where, the singleton is taken n times. Since the singleton mapping is injective at each step, the overall map i is injective. Thus, we may identify each $e \in E_0$ with its image $i(e)$ in $\mathcal{P}^n(E_0)$.

Now, let

$$V = i(E_0) \subseteq \mathcal{P}^n(E_0) \tag{92}$$

and define the set of n-superfacts by

$$\tau_0^{(n)} = \{ r(i(e_1), \ldots, i(e_{|r|})) \mid r(e_1, \ldots, e_{|r|}) \in \tau_0 \}. \tag{93}$$

Similarly, relations in R_0 are embedded into $\mathcal{P}^n(R_0)$. With these definitions, the structure

$$\mathcal{KH}^{(n)} = (V, R', \tau_0^{(n)}) \tag{94}$$

contains an isomorphic copy of the original knowledge HyperGraph $\mathcal{H} = (E_0, R_0, \tau_0)$. Therefore, every knowledge HyperGraph is a special case (or can be embedded as a substructure) of a knowledge n-SuperHyperGraph.

3.2 Multimodal n-SuperHyperGraph

The definition of the multimodal n-SuperHyperGraph is given as follows.

Definition21 Multimodal n-SuperHyperGraph. Let V_0 be a finite base set. For an integer $n \geq 0$, denote by $\mathcal{P}^n(V_0)$ the nth powerset of V_0, defined recursively as

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{n+1}(V_0) = \mathcal{P}(\mathcal{P}^n(V_0)). \tag{95}$$

For each modality $m = 1, 2, \ldots, M$, let

$$G_m^{(n)} = (V, E_m, W_m) \tag{96}$$

be an n-SuperHyperGraph defined on a common vertex set

$$V \subseteq \mathcal{P}^n(V_0), \tag{97}$$

where,
- $E_m \subseteq \mathcal{P}^n(V_0)$ is a set of n-superedges (each being a nonempty subset of V; typically, $|e| \geq 2$ for all $e \in E_m$);
- $W_m : E_m \to \mathbb{R}^+$ is a function assigning a positive weight to each n-superedge in E_m.

Furthermore, let $\{\alpha_m\}_{m=1}^M$ be a set of combination weights satisfying

$$\alpha_m \geq 0 \quad \text{for all } m = 1, \ldots, M, \qquad \text{and} \qquad \sum_{m=1}^M \alpha_m = 1. \tag{98}$$

Then, the multimodal n-SuperHyperGraph is defined as the tuple

$$\mathcal{G}^{(n)} = \left(V, \{E_m\}_{m=1}^M, \{W_m\}_{m=1}^M, \{\alpha_m\}_{m=1}^M\right). \tag{99}$$

Notice that when $n = 0$ (so that $\mathcal{P}^0(V_0) = V_0$), the structure reduces to the standard definition of a multimodal hypergraph.

Example7 Multimodal 1-SuperHyperGraph. Let the base set be

$$V_0 = \{a, b, c\}. \tag{100}$$

Then the first powerset is

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}. \tag{101}$$

Define the common vertex set by taking the nonempty subsets:

$$V = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}. \tag{102}$$

Assume there are two modalities ($M = 2$). For modality $m = 1$, let

$$E_1 = \left\{\{\{a\}, \{b\}\}, \{\{b\}, \{c\}\}\right\}, \tag{103}$$

with weight function $W_1(e) = 1$ for every $e \in E_1$. For modality $m = 2$, let

$$E_2 = \left\{\{\{a\}, \{c\}\}, \{\{a, b\}, \{a, c\}\}\right\}, \tag{104}$$

with weight function $W_2(e) = 2$ for every $e \in E_2$. Finally, choose combination weights $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$. Then the multimodal 1-SuperHyperGraph is

$$\mathcal{G}^{(1)} = \left(V, \{E_1, E_2\}, \{W_1, W_2\}, \{\alpha_1, \alpha_2\}\right). \tag{105}$$

Example8 Multimodal 2-SuperHyperGraph for a Smart Campus. We consider a small smart campus equipped with ambient sensors, where sensors are grouped into rooms, and rooms are further grouped into buildings. We model building-level multimodal relationships (e.g., energy usage similarity and security alerts) by constructing a multimodal 2-SuperHyperGraph $\mathcal{G}^{(2)}$.

Base Set V_0. Let

$$V_0 = \{s_1, s_2, s_3, s_4, s_5, s_6\} \tag{106}$$

be six ambient sensors deployed throughout the campus.

First Powerset $\mathcal{P}^1(V_0)$ (Rooms). The elements of $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ include all subsets of sensors. We select three subsets to represent three rooms:

$$R_1 = \{s_1, s_2\}, \quad R_2 = \{s_3, s_4\}, \quad R_3 = \{s_5, s_6\}. \tag{107}$$

Each R_i corresponds to a room with two sensors.

Second Powerset $\mathcal{P}^2(V_0)$ (Buildings). By definition,

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big), \tag{108}$$

whose elements are sets of subsets of V_0. We define three elements of $\mathcal{P}^2(V_0)$ to represent three buildings:

$$\begin{aligned}
B_1 &= \{R_1, R_2\}, \\
B_2 &= \{R_2, R_3\}, \\
B_3 &= \{R_1, R_3\}.
\end{aligned} \tag{109}$$

Thus:

$$\begin{aligned}
B_1 &= \{\{s_1, s_2\}, \{s_3, s_4\}\}, \\
B_2 &= \{\{s_3, s_4\}, \{s_5, s_6\}\}, \\
B_3 &= \{\{s_1, s_2\}, \{s_5, s_6\}\}.
\end{aligned} \tag{110}$$

Each $B_i \in \mathcal{P}^2(V_0)$ is called a 2-supervertex (i.e., a building containing two rooms).

Common Vertex Set V. We set

$$V = \{B_1, B_2, B_3\}. \tag{111}$$

All three buildings use the same "rooms" R_1, R_2, R_3 drawn from $\mathcal{P}^1(V_0)$.

Two Modalities ($M = 2$).

Modality 1: Energy-Usage Similarity. We observe that during peak hours, Buildings B_1 and B_2 have correlated energy usage patterns (e.g., similar heating/AC loads), and likewise B_2 and B_3 share similarity (e.g., similar lighting usage). We define the set of 2-superedges:

$$E_1 = \big\{e_{1,1} = \{B_1, B_2\},\ e_{1,2} = \{B_2, B_3\}\big\}, \tag{112}$$

where, each hyperedge $e_{1,i} \subseteq V$ has cardinality $|e_{1,i}| = 2$. Assign weights according to the measured correlation scores:

$$W_1(e_{1,1}) = 0.85, \quad W_1(e_{1,2}) = 0.78. \tag{113}$$

Here $W_1 : E_1 \to \mathbb{R}^+$ quantifies the degree of energy-usage similarity between the two buildings in each hyperedge.

Modality 2: Security-Alert Co-Occurrence. Over a monitoring period, security alerts (e.g., unauthorized door access) occurred concurrently in B_1 and B_3, but not in B_2. We define:

$$E_2 = \big\{e_{2,1} = \{B_1, B_3\}\big\}, \tag{114}$$

with weight function

$$W_2(e_{2,1}) = 1.0, \tag{115}$$

reflecting a strong co-occurrence of security events in those two buildings.

Combination Weights. To integrate these two modalities, choose:

$$\alpha_1 = 0.6, \quad \alpha_2 = 0.4, \quad \alpha_1 + \alpha_2 = 1. \tag{116}$$

These weights indicate that energy-usage similarity ($\alpha_1 = 0.6$) is considered somewhat more important than security-alert co-occurrence ($\alpha_2 = 0.4$) in downstream tasks.

Multimodal 2-SuperHyperGraph. Putting everything together, the multimodal 2-SuperHyperGraph is

$$\mathcal{G}^{(2)} = \Big(V,\ \{E_1, E_2\},\ \{W_1, W_2\},\ \{\alpha_1, \alpha_2\}\Big). \tag{117}$$

Flattening Function F. Recall $F : \mathcal{P}^2(V_0) \to \mathcal{P}(V_0)$ "flattens" each building to the union of its rooms (subsets of sensors):

$$\begin{aligned}
F(B_1) &= R_1 \cup R_2 = \{s_1, s_2, s_3, s_4\}, \\
F(B_2) &= R_2 \cup R_3 = \{s_3, s_4, s_5, s_6\}, \\
F(B_3) &= R_1 \cup R_3 = \{s_1, s_2, s_5, s_6\}.
\end{aligned} \tag{118}$$

This ensures that each 2-superedge $e \subseteq V$ corresponds to overlapping or adjacent sensor sets at the ground level when appropriate (e.g., B_1 and B_2 share R_2).

Interpretation.

- Modality 1 (Energy-Usage). $e_{1,1} = \{B_1, B_2\}$ indicates that Buildings B_1 and B_2 exhibit similar energy-usage patterns during peak hours. Flattened, they share room R_2 with sensors $\{s_3, s_4\}$. $e_{1,2} = \{B_2, B_3\}$ captures similarity between B_2 and B_3, which both draw significant lighting power from rooms R_2 or R_3 in evening.
- Modality 2 (Security-Alert). $e_{2,1} = \{B_1, B_3\}$ shows that security alerts co-occur in Buildings B_1 and B_3, perhaps because both have entrances monitored by shared camera arrays (flattened sets $\{s_1, s_2, s_5, s_6\}$).
- By assigning $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$, we prioritize energy-usage similarity over security co-occurrence when, for example, performing community detection or anomaly detection on the campus.

Thus, $\mathcal{G}^{(2)} = (V, \{E_1, E_2\}, \{W_1, W_2\}, \{\alpha_1, \alpha_2\})$ provides a detailed, higher-order model of building-level relationships based on two modalities in a finite smart-campus setting.

Theorem3. Every multimodal hypergraph is a special case of a multimodal n-SuperHyperGraph. In particular, for $n = 0$,

$$\mathcal{G}^{(0)} = \left(V,\ \{E_m\}_{m=1}^{M},\ \{W_m\}_{m=1}^{M},\ \{\alpha_m\}_{m=1}^{M}\right) \tag{119}$$

is exactly the classical multimodal hypergraph. Moreover, for any $n \geq 1$, there exists a canonical flattening function

$$F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0) \tag{120}$$

defined recursively by

$$F(x) = x \quad \text{for } x \in V_0, \quad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0) \text{ with } k \geq 1, \tag{121}$$

such that applying F to every n-superedge in each modality yields a multimodal hypergraph

$$\mathcal{G}' = \left(V_0,\ \{E_m'\}_{m=1}^{M},\ \{W_m\}_{m=1}^{M},\ \{\alpha_m\}_{m=1}^{M}\right), \tag{122}$$

where,

$$E_m' = \{F(e) \mid e \in E_m\} \subseteq \mathcal{P}(V_0). \tag{123}$$

Proof. There are two cases.

Case 1: $n = 0$. By definition, $\mathcal{P}^0(V_0) = V_0$. Hence, the vertex set $V \subseteq \mathcal{P}^0(V_0)$ is exactly a subset of V_0, and each hyperedge $e \in E_m$ is a nonempty subset of V_0. Thus, the multimodal 0-SuperHyperGraph

$$\mathcal{G}^{(0)} = \left(V,\ \{E_m\}_{m=1}^{M},\ \{W_m\}_{m=1}^{M},\ \{\alpha_m\}_{m=1}^{M}\right) \tag{124}$$

is identical to the classical multimodal HyperGraph definition.

Case 2: $n \geq 1$. For any $n \geq 1$, each vertex or hyperedge is an element of $\mathcal{P}^n(V_0)$. Define the flattening function $F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0)$ recursively by:

$$F(x) = x \quad \text{if } x \in V_0, \qquad F(A) = \bigcup_{a \in A} F(a) \quad \text{if } A \in \mathcal{P}^k(V_0),\ k \geq 1. \tag{125}$$

For each modality m, transform its hyperedge set by setting

$$E_m' = \{F(e) \mid e \in E_m\}. \tag{126}$$

Since $F(e)$ is a subset of V_0 for every $e \in E_m$, it follows that

$$E_m' \subseteq \mathcal{P}(V_0). \tag{127}$$

Thus, by taking V_0 as the new vertex set and E_m' as the hyperedge set for each modality m, we obtain a multimodal hypergraph

$$\mathcal{G}' = \left(V_0,\ \{E_m'\}_{m=1}^{M},\ \{W_m\}_{m=1}^{M},\ \{\alpha_m\}_{m=1}^{M}\right). \tag{128}$$

This shows that any multimodal n-SuperHyperGraph can be transformed (or "flattened") into a classical multimodal hypergraph, hence generalizing the latter.

3.3 Lattice n-SuperHyperGraph

The definition of the Lattice n-SuperHyperGraph is given as follows.

Definition22 Lattice n-SuperHyperGraph. Let V_0 be a finite base set representing elementary lattice cells (e.g., grid cells in a geometric layout). For any integer $n \geq 0$, define the nth powerset of V_0 recursively by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\big(\mathcal{P}^n(V_0)\big). \tag{129}$$

A Lattice n-SuperHyperGraph is a heterogeneous graph

$$\mathcal{G}^{(n)} = \big(V_c^{(n)}, V_n^{(n)}, A^{(n)}, H^{(n)}\big), \tag{130}$$

where,

- $V_c^{(n)} \subseteq \mathcal{P}^n(V_0)$ is the set of n-super lattice vertices (each representing a higher-order lattice cell, generalizing a standard grid cell),
- $V_n^{(n)} \subseteq \mathcal{P}^n(V_0)$ is the set of n-super hyperedge nodes (each representing a grouping of lattice vertices—for example, a generalized G-net),
- The lattice adjacency matrix $A^{(n)} \in \{0,1\}^{|V_c^{(n)}| \times |V_c^{(n)}|}$ is defined by

$$A_{ij}^{(n)} = \begin{cases} 1, & \text{if the flattened images } F(v_i) \text{ and } F(v_j) \text{ are adjacent in the base lattice,} \\ 0, & \text{otherwise,} \end{cases} \tag{131}$$

for $v_i, v_j \in V_c^{(n)}$,

- The incidence matrix $H^{(n)} \in \{0,1\}^{|V_c^{(n)}| \times |V_n^{(n)}|}$ is defined by

$$H_{ij}^{(n)} = \begin{cases} 1, & \text{if } F(v_i) \subseteq F(u_j), \\ 0, & \text{otherwise,} \end{cases} \tag{132}$$

for $v_i \in V_c^{(n)}$ and $u_j \in V_n^{(n)}$.

Here, the flattening function

$$F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0) \tag{133}$$

is defined recursively by

$$F(x) = x \quad \text{for } x \in V_0, \qquad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0),\ k \geq 1. \tag{134}$$

Example9 Lattice 1-SuperHyperGraph. Let the base set be

$$V_0 = \{v_1, v_2, v_3\}, \tag{135}$$

which represent three adjacent grid cells in a layout.

Then the first powerset is

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}. \tag{136}$$

For the lattice vertices, choose the singletons:

$$V_c^{(1)} = \{\{v_1\}, \{v_2\}, \{v_3\}\}. \tag{137}$$

For the hyperedge nodes, define groups (e.g., adjacent cells) by

$$V_n^{(1)} = \{\{v_1, v_2\}, \{v_2, v_3\}\}. \tag{138}$$

Using the flattening function F, note that

$$F(\{v_1\}) = \{v_1\}, \quad F(\{v_1, v_2\}) = \{v_1, v_2\}. \tag{139}$$

Assume in the base lattice that v_1 is adjacent to v_2, v_2 is adjacent to v_3, and v_1 is not adjacent to v_3. Then the lattice adjacency matrix for $V_c^{(1)}$ is

$$A^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{140}$$

The incidence matrix $H^{(1)}$ is defined by membership:

$$H^{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{141}$$

since $\{v_1\} \subseteq \{v_1, v_2\}$, $\{v_2\}$ is in both $\{v_1, v_2\}$ and $\{v_2, v_3\}$, and $\{v_3\} \subseteq \{v_2, v_3\}$.

Thus, the lattice 1-SuperHyperGraph is given by

$$\mathcal{G}^{(1)} = \left(V_c^{(1)}, V_n^{(1)}, A^{(1)}, H^{(1)}\right). \tag{142}$$

Example10 Lattice 2-SuperHyperGraph for a 2 × 2 Chip Grid. We illustrate a lattice 2-SuperHyperGraph $\mathcal{G}^{(2)} = \left(V_c^{(2)}, V_n^{(2)}, A^{(2)}, H^{(2)}\right)$ using a simple 2×2 grid of G-cells on a chip.

Base Set V_0. Let the base set of elementary lattice cells be

$$V_0 = \{ c_1, c_2, c_3, c_4 \}, \tag{143}$$

arranged in a 2×2 layout as:

$$\begin{array}{cc} c_1 & c_2 \\ c_3 & c_4 \end{array} \tag{144}$$

with the usual adjacency: c_1 adjacent to c_2 and c_3, c_2 adjacent to c_1 and c_4, c_3 adjacent to c_1 and c_4, c_4 adjacent to c_2 and c_3.

First Powerset $\mathcal{P}(V_0)$. The powerset $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ consists of all subsets of V_0. For instance,

$$\{ c_1 \}, \{ c_2 \}, \{ c_1, c_2 \}, \{ c_3, c_4 \}, \{ c_1, c_2, c_3, c_4 \}, \ldots \tag{145}$$

However, we will use only a few of these in the construction that follows.

Second Powerset $\mathcal{P}^2(V_0)$. By definition,

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big), \tag{146}$$

whose elements are sets of subsets of V_0. Two relevant elements in $\mathcal{P}^2(V_0)$ are:

$$x_1 = \{\{c_1\}, \{c_2\}\}, \qquad x_2 = \{\{c_3\}, \{c_4\}\}. \tag{147}$$

Each x_i is itself a set whose members lie in $\mathcal{P}^1(V_0)$.

Set of 2-Super Lattice Vertices $V_c^{(2)}$. We choose

$$V_c^{(2)} = \{ x_1, x_2 \}, \tag{148}$$

where,

$$x_1 = \{\{c_1\}, \{c_2\}\}, \quad x_2 = \{\{c_3\}, \{c_4\}\}. \tag{149}$$

These are our 2-super lattice vertices. Each $x_i \in \mathcal{P}^2(V_0)$.

Set of 2-Super Hyperedge Nodes $V_n^{(2)}$. We define one higher-order grouping:

$$u_1 = \{\{c_1, c_2\}, \{c_3, c_4\}\}, \tag{150}$$

so that

$$V_n^{(2)} = \{ u_1 \}, \quad u_1 \in \mathcal{P}^2(V_0). \tag{151}$$

Flattening Function F. Recall that $F : \mathcal{P}^2(V_0) \to \mathcal{P}(V_0)$ is defined by

$$F(x) = \bigcup_{a \in x} F(a), \quad F(a) = a \text{ for } a \in \mathcal{P}^1(V_0). \tag{152}$$

hence,

$$\begin{aligned} F(x_1) &= F\big(\{\{c_1\}, \{c_2\}\}\big) = \{c_1\} \cup \{c_2\} = \{ c_1, c_2 \}, \\ F(x_2) &= \{c_3, c_4\}, \\ F(u_1) &= F\big(\{\{c_1, c_2\}, \{c_3, c_4\}\}\big) = \{c_1, c_2\} \cup \{c_3, c_4\} = \{ c_1, c_2, c_3, c_4 \}. \end{aligned} \tag{153}$$

Lattice Adjacency Matrix $A^{(2)}$**.** We declare two 2-super vertices $x_i, x_j \in V_c^{(2)}$ to be adjacent if $F(x_i)$ and $F(x_j)$ contain at least one pair of adjacent base cells in V_0. In our case:

$$F(x_1) = \{c_1, c_2\}, \quad F(x_2) = \{c_3, c_4\}. \tag{154}$$

Since c_2 is adjacent to c_4 in the base 2×2 grid, we set:

$$A_{1,2}^{(2)} = 1, \quad A_{2,1}^{(2)} = 1, \tag{155}$$

and, of course,

$$A_{1,1}^{(2)} = 0, \quad A_{2,2}^{(2)} = 0. \tag{156}$$

Hence, in matrix form (indexing x_1 as row/column 1 and x_2 as row/column 2):

$$A^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{157}$$

Incidence Matrix $H^{(2)}$**.** We link each 2-super vertex $x_i \in V_c^{(2)}$ to each 2-super hyperedge node $u_j \in V_n^{(2)}$ whenever $F(x_i) \subseteq F(u_j)$. Here:

$$F(x_1) = \{c_1, c_2\} \subseteq \{c_1, c_2, c_3, c_4\} = F(u_1), \quad F(x_2) = \{c_3, c_4\} \subseteq F(u_1). \tag{158}$$

Thus,

$$H_{1,1}^{(2)} = 1, \quad H_{2,1}^{(2)} = 1. \tag{159}$$

In matrix form (rows indexed by x_1, x_2; column indexed by u_1):

$$H^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{160}$$

Interpretation.
- The set $V_c^{(2)} = \{x_1, x_2\}$ consists of two 2-super vertices: x_1 flattens to $\{c_1, c_2\}$ (the top row), and x_2 flattens to $\{c_3, c_4\}$ (the bottom row).
- The adjacency $A_{1,2}^{(2)} = 1$ arises because the flattened top row $\{c_1, c_2\}$ shares a border with the flattened bottom row $\{c_3, c_4\}$ at $c_2 \sim c_4$.
- The set $V_n^{(2)} = \{u_1\}$ represents the entire 2×2 grid as a single hyperedge, since u_1 flattens to all four cells $\{c_1, c_2, c_3, c_4\}$.
- The incidence matrix $H^{(2)}$ has both entries equal to 1, indicating that each 2-super vertex (x_1 or x_2) is contained within the hyperedge u_1.

Thus, $\mathcal{G}^{(2)} = \left(V_c^{(2)}, V_n^{(2)}, A^{(2)}, H^{(2)}\right)$ provides a higher-order model of the original 2×2 grid. It captures both the local adjacency of super-cells along the row boundary and the global grouping of all cells by the single 2-super hyperedge u_1. Such a structure can be useful in applications like hierarchical routing or multi-scale optimization on chip layouts.

Theorem4. Let $\mathcal{G} = \left(V_c, V_n, A, H\right)$ be a classical lattice hypergraph defined on a base set V_0, where

$$V_c \subseteq V_0 \quad \text{and} \quad V_n \subseteq \mathcal{P}(V_0). \tag{161}$$

Then, for any $n \geq 0$, there exists a lattice n-SuperHyperGraph

$$\mathcal{G}^{(n)} = \left(V_c^{(n)}, V_n^{(n)}, A^{(n)}, H^{(n)}\right) \tag{162}$$

and a flattening function $F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0)$ such that

$$\{F(v) : v \in V_c^{(n)}\} = V_c \quad \text{and} \quad \{F(u) : u \in V_n^{(n)}\} = V_n, \tag{163}$$

and the corresponding flattened adjacency and incidence matrices equal A and H, respectively. In other words, the classical lattice hypergraph is a special case (obtained by flattening) of the lattice n-SuperHyperGraph.

Proof. We prove the claim by constructing a flattening function F and showing that the classical lattice hypergraph is recovered.

Case $n = 0$**:** When $n = 0$, we have $\mathcal{P}^0(V_0) = V_0$. Thus, $V_c^{(0)}$ and $V_n^{(0)}$ are subsets of V_0 and $\mathcal{P}(V_0)$, respectively. In this case, the lattice 0-SuperHyperGraph

$$\mathcal{G}^{(0)} = \left(V_c^{(0)}, V_n^{(0)}, A^{(0)}, H^{(0)}\right) \tag{164}$$

is exactly the classical lattice hypergraph.

Case $n \geq 1$**:** Define the flattening function $F : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0)$ recursively by

$$F(x) = x \quad \text{for } x \in V_0, \qquad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0), \; k \geq 1. \tag{165}$$

Now, suppose that the lattice n-SuperHyperGraph

$$\mathcal{G}^{(n)} = \left(V_c^{(n)}, \; V_n^{(n)}, \; A^{(n)}, \; H^{(n)} \right) \tag{166}$$

has been constructed so that its vertex set $V_c^{(n)}$ and hyperedge node set $V_n^{(n)}$ satisfy

$$\{F(v) : v \in V_c^{(n)}\} = V_c \quad \text{and} \quad \{F(u) : u \in V_n^{(n)}\} = V_n. \tag{167}$$

Moreover, by defining the n-level lattice adjacency matrix $A^{(n)}$ and incidence matrix $H^{(n)}$ using the flattened representations (i.e., two n-supervertices $v_i, v_j \in V_c^{(n)}$ are declared adjacent if and only if their flattenings $F(v_i)$ and $F(v_j)$ are adjacent in the base lattice; similarly for $H^{(n)}$), the flattened matrices satisfy

$$A_{ij} = A_{ij}^{(n)} \quad \text{and} \quad H_{ij} = H_{ij}^{(n)}. \tag{168}$$

Thus, by applying F to all elements, we recover the classical lattice hypergraph

$$\mathcal{G}' = \left(V_c, \; V_n, \; A, \; H \right). \tag{169}$$

This demonstrates that every classical lattice hypergraph is obtained as the flattening of a lattice n-SuperHyperGraph.

3.4 Hyperbolic n-SuperHyperGraph

The definition of the hyperbolic n-SuperHyperGraph is given as follows.

Definition23 Hyperbolic n-SuperHyperGraph. Let V_0 be a finite base set. For a nonnegative integer n, define the nth powerset of V_0 recursively by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\left(\mathcal{P}^n(V_0)\right). \tag{170}$$

Let $\mathbb{H}^d$ denote a d-dimensional hyperbolic space (i.e. a complete Riemannian manifold with constant negative curvature $-\kappa$ for some $\kappa > 0$). A hyperbolic n-SuperHyperGraph is defined as the triple

$$\mathcal{H}_{\mathbb{H}}^{(n)} = (V, E, \phi), \tag{171}$$

where,
- $V \subseteq \mathcal{P}^n(V_0)$ is a set of n-supervertices,
- $E \subseteq \{ e \subseteq V \mid |e| \geq 2 \}$ is a collection of n-superedges,
- $\phi : V \to \mathbb{H}^d$ is an injective mapping (embedding) that assigns to each n-supervertex a unique point in $\mathbb{H}^d$.

Optionally, for each hyperedge $e \in E$, one may define an aggregated representation $\mu(e)$ (e.g., the hyperbolic Fréchet mean of $\{\phi(v) : v \in e\}$) to capture the collective geometry of the vertices in e.

Example11 Hyperbolic 1-SuperHyperGraph. Let the base set be

$$V_0 = \{a, b, c\}. \tag{172}$$

Then the first powerset is

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}. \tag{173}$$

Define the set of 1-supervertices by taking the nonempty subsets:

$$V = \{\{a\}, \{b\}, \{c\}, \{a,b\}\}. \tag{174}$$

Consider the collection of 1-superedges:

$$E = \Big\{ \{\{a\}, \{b\}\}, \quad \{\{b\}, \{a,b\}\} \Big\}. \tag{175}$$

Let $\mathbb{H}^2$ denote the hyperbolic plane (e.g., in the Poincaré disk model). Define an embedding $\phi : V \to \mathbb{H}^2$ by assigning four distinct points in $\mathbb{H}^2$ to the vertices $\{a\}$, $\{b\}$, $\{c\}$, and $\{a,b\}$. Then the triple

$$\mathcal{H}_{\mathbb{H}}^{(1)} = (V, E, \phi) \tag{176}$$

forms a concrete example of a hyperbolic 1-SuperHyperGraph.

Example12 Hyperbolic 2-SuperHyperGraph for Company Organization. We model a company's organizational hierarchy and inter-departmental projects as a hyperbolic 2-SuperHyperGraph $\mathcal{H}_{\mathbb{H}}^{(2)} = (V, E, \phi)$. At the base level, individual employees are grouped into teams, teams into departments, and departments collaborate on projects. Hyperbolic geometry naturally encodes the hierarchical structure, with more specific units near the boundary and broader units near the center of $\mathbb{H}^2$.

Base Set V_0. Let

$$V_0 = \{ \text{Alice, Bob, Carol, Dave, Eve, Frank} \} \tag{177}$$

be six employees.

First Powerset $\mathcal{P}^1(V_0)$ (Teams). We form three teams as subsets of V_0:

$$
\begin{aligned}
T_1 &= \{\text{Alice, Bob}\}, \\
T_2 &= \{\text{Carol, Dave}\}, \\
T_3 &= \{\text{Eve, Frank}\}.
\end{aligned}
\tag{178}
$$

Each $T_i \in \mathcal{P}^1(V_0)$ represents a "team" of two employees.

Second Powerset $\mathcal{P}^2(V_0)$ (Departments). $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ consists of sets of teams. We select:

$$
\begin{aligned}
D_1 &= \{ T_1, T_2 \}, \quad \text{(Department 1: Teams 1 and 2)}, \\
D_2 &= \{ T_2, T_3 \}, \quad \text{(Department 2: Teams 2 and 3)}, \\
O &= \{ T_1, T_2, T_3 \}, \quad \text{(Company: all three teams)}.
\end{aligned}
\tag{179}
$$

Thus $D_1, D_2, O \in \mathcal{P}^2(V_0)$. We interpret D_1, D_2 as two departments, and O as the entire organization.

Set of 2-SuperVertices V.

$$V = \{ D_1, D_2, O \}, \tag{180}$$

where, each element is an n-supervertex with $n = 2$.

Hyperedges E. We model two types of inter-departmental collaborations:

- Project Alpha involves Departments D_1 and D_2:

$$e_1 = \{ D_1, D_2 \}. \tag{181}$$

- All-Hands Summit involves Department D_1 and the entire organization O:

$$e_2 = \{ D_1, O \}, \quad e_3 = \{ D_2, O \}. \tag{182}$$

Thus,

$$E = \{ e_1, e_2, e_3 \}, \quad e_1 = \{D_1, D_2\}, \quad e_2 = \{D_1, O\}, \quad e_3 = \{D_2, O\}. \tag{183}$$

Each $e \subseteq V$ has $|e| \geq 2$.

Hyperbolic Embedding ϕ. Choose the Poincaré disk model of the hyperbolic plane $\mathbb{H}^2$. We embed:

$$
\begin{aligned}
\phi(D_1) &= (r = 0.8,\ \theta = 0°), \\
\phi(D_2) &= (r = 0.8,\ \theta = 180°), \\
\phi(O) &= (r = 0.2,\ \theta = 0°).
\end{aligned}
\tag{184}
$$

That is, D_1 and D_2 lie near the boundary at opposite angles, reflecting that they are distinct, specialized departments. The company node O is placed near the center ($r = 0.2$), indicating a more general, higher-level category.

Hyperbolic Distances. For any two vertices $u, v \in V$,

$$d_{\mathbb{H}}(\phi(u), \phi(v)) = \operatorname{arcosh}\Big(1 + 2\,\frac{\|\phi(u) - \phi(v)\|^2}{(1 - \|\phi(u)\|^2)\,(1 - \|\phi(v)\|^2)}\Big). \tag{185}$$

For instance,

$$d_{\mathbb{H}}\big(\phi(D_1),\ \phi(D_2)\big) \text{ is large (two departments far apart near boundary)}, \tag{186}$$

whereas

$$d_{\mathbb{H}}\big(\phi(D_1),\ \phi(O)\big) \text{ is moderate (department to company center)}. \tag{187}$$

Hyperedge Representations $\mu(e)$. Optionally, for each $e \in E$, we can define

$$\mu(e) = \arg\min_{z \in \mathbb{H}^2} \sum_{v \in e} d_{\mathbb{H}}\big(z,\ \phi(v)\big)^2 \tag{188}$$

as the hyperbolic Fréchet mean. Concretely:

- $\mu(e_1)$ (Project Alpha) lies along the geodesic midpoint between $\phi(D_1)$ and $\phi(D_2)$, at moderate radius $r \approx 0.8$.
- $\mu(e_2)$ (D$_1$ & O) lies between $\phi(D_1)$ and $\phi(O)$, closer to O since O is at small r.
- $\mu(e_3)$ (D$_2$ & O) similarly lies between $\phi(D_2)$ and $\phi(O)$.

Real-World Interpretation.
- $D_1 = \{ T_1, T_2 \}$ is Department 1 (e.g., "Engineering & R&D"), flattening to $T_1 \cup T_2 = \{$Alice, Bob, Carol, Dave$\}$.
- $D_2 = \{ T_2, T_3 \}$ is Department 2 (e.g., "R&D & Operations"), flattening to $\{$Carol, Dave, Eve, Frank$\}$.
- $O = \{ T_1, T_2, T_3 \}$ is the entire company, flattening to all six employees $\{$Alice, ..., Frank$\}$.
- **Project Alpha** (e_1) is a collaboration between Department 1 and Department 2. In hyperbolic space, D_1 and D_2 are near the boundary in opposite directions, reflecting distinct but interconnected subunits.
- **All-Hands Summit** comprises both Department 1 and the whole company: $e_2 = \{ D_1, O \}$. Its Fréchet mean $\mu(e_2)$ lies closer to the center, indicating a more general event. Similarly, $e_3 = \{ D_2, O \}$ captures Department 2 joining the company event.
- Embedding in $\mathbb{H}^2$ encodes the hierarchy: specialized departments ($r = 0.8$) are near the boundary, while the overall company ($r = 0.2$) is near the center. Distances in $\mathbb{H}^2$ correlate with organizational separation—departments that share more teams (e.g., D_1 and D_2 both include T_2) still appear relatively far because they reside near opposite angles.

Hence, $\mathcal{H}_{\mathbb{H}}^{(2)} = (V, E, \phi)$ provides a higher-order, hyperbolic representation of a company's team–department–organization hierarchy and their collaborative projects.

Theorem5. Every classical hyperbolic hypergraph is a special case of a hyperbolic n-SuperHyperGraph. In particular, let

$$\mathcal{H}_{\mathbb{H}} = (V_0, E_0, \phi_0) \tag{189}$$

be a classical hyperbolic hypergraph with $V_0 \subseteq V_0$, $E_0 \subseteq \{ e \subseteq V_0 \mid |e| \geq 2 \}$, and $\phi_0 : V_0 \to \mathbb{H}^d$ an injective embedding. Then, for any $n \geq 0$, there exists a hyperbolic n-SuperHyperGraph

$$\mathcal{H}_{\mathbb{H}}^{(n)} = (V, E, \phi) \tag{190}$$

and a flattening function

$$F : \mathcal{P}^n(V_0) \to V_0, \tag{191}$$

defined recursively by

$$F(x) = x \quad \text{for } x \in V_0, \qquad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0), \ k \geq 1, \tag{192}$$

such that

$$F(V) = V_0, \quad \{F(e) : e \in E\} = E_0, \quad \text{and} \quad \phi = \phi_0 \circ F. \tag{193}$$

Proof. We consider two cases.

Case 1: $n = 0$. Since $\mathcal{P}^0(V_0) = V_0$, a hyperbolic 0-SuperHyperGraph is given by

$$\mathcal{H}_{\mathbb{H}}^{(0)} = (V_0, E_0, \phi_0), \tag{194}$$

which is exactly the classical hyperbolic hypergraph $\mathcal{H}_{\mathbb{H}}$.

Case 2: $n \geq 1$. Define the flattening function $F : \mathcal{P}^n(V_0) \to V_0$ recursively by

$$F(x) = x \quad \text{for } x \in V_0, \qquad F(A) = \bigcup_{a \in A} F(a) \quad \text{for } A \in \mathcal{P}^k(V_0), \ k \geq 1. \tag{195}$$

Now, construct the hyperbolic n-SuperHyperGraph $\mathcal{H}_{\mathbb{H}}^{(n)} = (V, E, \phi)$ by choosing

$$V \subseteq \mathcal{P}^n(V_0) \tag{196}$$

such that $F(V) = V_0$ (i.e., for every $v \in V_0$ there is some $u \in V$ with $F(u) = v$), and choose

$$E \subseteq \{ e \subseteq V \mid |e| \geq 2 \} \tag{197}$$

so that

$$\{F(e) : e \in E\} = E_0. \tag{198}$$

Define the embedding $\phi : V \to \mathbb{H}^d$ by

$$\phi(u) = \phi_0(F(u)) \quad \text{for each } u \in V. \tag{199}$$

Since ϕ_0 is injective and F maps each $u \in V$ to an element of V_0, the composed mapping $\phi = \phi_0 \circ F$ is injective. By construction, flattening the vertices and hyperedges of $\mathcal{H}_{\mathbb{H}}^{(n)}$ via F recovers the classical hyperbolic hypergraph $\mathcal{H}_{\mathbb{H}}$. This shows that every hyperbolic hypergraph is a special case (obtained by flattening) of a hyperbolic n-SuperHyperGraph.

4 Conclusion and Future Work

In this paper, we extended Knowledge Hypergraphs, Multimodal Hypergraphs, Lattice Hypergraphs, and Hyperbolic Hypergraphs using the SuperHyperGraph framework, introducing Knowledge SuperHypergraphs, Multimodal SuperHypergraphs, Lattice SuperHypergraphs, and Hyperbolic SuperHypergraphs.

One potential direction is to further generalize the graph classes introduced here by incorporating elements from various advanced set theories, including Fuzzy Sets, Intuitionistic Fuzzy Sets, Soft Sets, Hypersoft sets, Neutrosophic Sets, Rough Sets, Plithogenic Sets, and Hyperrough Sets. Developing graph structures that integrate these concepts would provide new theoretical insights and broaden their range of applications. Furthermore, exploring real-world applications of these extended graph models could open up exciting avenues for future research.

Data Availability

Not applicable.

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Conflicts of Interest

The author declares no conflict of interest.

References

[1] C. Berge, *Hypergraphs: Combinatorics of Finite Sets*. Elsevier, 1984, vol. 45.

[2] D. Cai, M. Song, C. Sun, B. Zhang, S. Hong, and H. Li, "Hypergraph structure learning for hypergraph neural networks," in *Thirty-First International Joint Conference on Artificial Intelligence*, 2022, pp. 1923–1929.

[3] Y. Gao, Z. Zhang, H. Lin, X. Zhao, S. Du, and C. Zou, "Hypergraph learning: Methods and practices," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 44, no. 5, pp. 2548–2566, 2020. https://doi.org/10.1109/TPAMI.2020.3039374

[4] Y. Feng, H. You, Z. Zhang, R. Ji, and Y. Gao, "Hypergraph neural networks," in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 33, no. 1, 2019, pp. 3558–3565. https://doi.org/10.1609/aaai.v33i01.33013558

[5] B. K. Jose and Z. Tuza, "Hypergraph domination and strong independence," *Appl. Anal. Discrete Math.*, vol. 3, no. 2, pp. 347–358, 2009. https://doi.org/10.2298/AADM0902347J

[6] B. Fatemi, P. Taslakian, D. Vazquez, and D. Poole, "Knowledge hypergraphs: Prediction beyond binary relations," *arXiv preprint arXiv:1906.00137*, 2019. https://doi.org/10.48550/arXiv.1906.00137

[7] B. P. Bhuyan, T. P. Singh, R. Tomar, and A. Ramdane-Cherif, "Nesykhg: Neuro-symbolic knowledge hypergraphs," *Procedia Comput. Sci.*, vol. 235, pp. 1278–1288, 2024. https://doi.org/10.1016/j.procs.2024.04.121

[8] D. López and A. Lozano, "Shortest hyperpaths in a multimodal hypergraph with real-time information on some transit lines," *Transp. Res. Part A: Policy Pract.*, vol. 137, pp. 541–559, 2020. https://doi.org/10.1016/j.tra.2019.09.020

[9] F. Yao, C. Tian, J. Liu, Z. Zhang, Q. Liu, L. Jin, S. Li, X. Li, and X. Sun, "Thinking like an expert: Multimodal hypergraph-of-thought (hot) reasoning to boost foundation modals," *arXiv preprint arXiv:2308.06207*, 2023. https://doi.org/10.48550/arXiv.2308.06207

[10] B. Wang, G. Shen, D. Li, J. Hao, W. Liu, Y. Huang, H. Wu, Y. Lin, G. Chen, and P. A. Heng, "Lhnn: Lattice hypergraph neural network for vlsi congestion prediction," in *Proceedings of the 59th ACM/IEEE Design Automation Conference*, 2022, pp. 1297–1302. https://doi.org/10.1145/3489517.35306

[11] M. G. Rawson, A. Myers, R. Green, M. Robinson, and C. Joslyn, "Formal concept lattice representations and algorithms for hypergraphs," *arXiv preprint arXiv:2307.11681*, 2023. https://doi.org/10.48550/arXiv.2307.11681

[12] E. Solomonik, "Provably efficient algorithms for numerical tensor algebra," Ph.D. dissertation, University of California, Berkeley, 2014.

[13] Y. Li, H. Chen, X. Sun, Z. Sun, L. Li, P. Cui, P. S. Yu, and G. Xu, "Hyperbolic hypergraphs for sequential recommendation," in *Proceedings of the 30th ACM International Conference on Information & Knowledge Management*, 2021, pp. 988–997. https://doi.org/10.1145/3459637.3482351

[14] S. Agarwal, R. Sawhney, M. Thakkar, P. Nakov, J. Han, and T. Derr, "Think: Temporal hypergraph hyperbolic network," in *2022 IEEE International Conference on Data Mining (ICDM)*, 2022, pp. 849–854.

[15] F. Smarandache, "Introduction to the n-superhypergraph-the most general form of graph today," *Neutrosophic Sets Syst.*, vol. 48, no. 1, p. 30, 2022.

[16] M. Hamidi, F. Smarandache, and E. Davneshvar, "Spectrum of superhypergraphs via flows," *J. Math.*, vol. 2022, no. 1, p. 9158912, 2022. https://doi.org/10.1155/2022/9158912

[17] M. Ghods, Z. Rostami, and F. Smarandache, "Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted super-hypertrees and several of their properties," *Neutrosophic Sets Syst.*, vol. 50, pp. 480–487, 2022.

[18] T. Fujita and F. Smarandache, "A concise study of some superhypergraph classes," *Neutrosophic Sets Syst.*, vol. 77, pp. 548–593, 2024.

[19] M. Hamidi, F. Smarandache, and M. Taghinezhad, "Decision making based on valued fuzzy superhypergraphs," *Comput. Model. Eng. Sci.*, vol. 138, no. 2, 2024. https://doi.org/10.32604/cmes.2023.030284

[20] E. M. C. Valencia, J. P. C. Vásquez, and F. Á. B. Lois, "Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs," *Neutrosophic Sets Syst.*, vol. 84, pp. 341–355, 2025. https://doi.org/10.5281/zenodo.15640723

[21] T. Fujita, "Modeling hierarchical systems in graph signal processing, electric circuits, and bond graphs via hypergraphs and superhypergraphs," *J. Eng. Res. Rep.*, vol. 27, no. 5, p. 542, 2025. https://doi.org/10.9734/jerr/2025/v27i51522

[22] R. Diestel, *Graph Theory*. Springer Nature, 2025, vol. 173.

[23] A. Bretto, *Hypergraph Theory: An Introduction*. Cham: Springer, 2013, vol. 1. https://doi.org/10.1007/978-3-319-00080-0

[24] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*. Springer Berlin, Heidelberg, 2003.

[25] F. Smarandache, "Foundation of superhyperstructure & neutrosophic superhyperstructure," *Neutrosophic Sets Syst.*, vol. 63, no. 1, p. 21, 2024.

[26] F. Smarandache, "The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures," *Sy. Assess. Eng. Manage.*, vol. 2, pp. 19–22, 2024. https://doi.org/10.61356/j.saem.2024.2436

[27] M. Jdid, F. Smarandache, and T. Fujita, "A linear mathematical model of the vocational training problem in a company using neutrosophic logic, hyperfunctions, and superhyperfunction," *Neutrosophic Sets Syst.*, vol. 87, pp. 1–11, 2025.

[28] F. Smarandache, *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[29] M. Novák, Š. Křehlík, and K. Ovaliadis, "Elements of hyperstructure theory in uwsn design and data aggregation," *Symmetry*, vol. 11, no. 6, p. 734, 2019. https://doi.org/10.3390/sym11060734

[30] S. A. Adebisi and A. P. Ajuebishi, "The order involving the neutrosophic hyperstructures, the construction and setting up of a typical neutrosophic group," *HyperSoft Set Methods Eng.*, vol. 3, pp. 26–31, 2025. https://doi.org/10.61356/j.hsse.2025.3426

[31] G. Amiri, R. Mousarezaei, and S. Rahnama, "Soft hyperstructures and their applications," *New Math. Nat. Comput.*, pp. 1–19, 2024. https://doi.org/10.1142/S1793005725500267

[32] T. Fujita, "Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems," in *Advancing Uncertain Combinatorics Through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, vol. 5. NSIA, 2025, p. 416.

[33] F. Smarandache, "Introduction to superhyperalgebra and neutrosophic superhyperalgebra," *J. Algebr. Hyperstruct. Log. Algebras*, vol. 3, no. 2, pp. 17–24, 2022. https://doi.org/10.52547/hatef.jahla.3.2.2

[34] S. Vougioukli, "Hyperoperations defined on sets of s-helix matrices," *J. Algebr. Hyperstruct. Log. Algebras*, vol. 1, no. 3, pp. 81–90, 2020. https://doi.org/10.29252/hatef.jahla.1.3.7

[35] T. Vougiouklis, *Hyperstructures and Their Representations*. Hadronic Press, 1994.

[36] N. B. Nalawade, M. S. Bapat, S. G. Jakkewad, G. A. Dhanorkar, and D. J. Bhosale, "Structural properties of zero-divisor hypergraph and superhypergraph over zn: Girth and helly property," *Panam. Math. J.*, vol. 35, no. 4S, p. 485, 2025.

[37] F. Smarandache, *Introduction to the n-SuperHyperGraph-the Most General Form of Graph Today*. Infinite Study, 2022.

[38] F. Smarandache, *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[39] T. Fujita, "An introduction and reexamination of molecular hypergraph and molecular n-superhypergraph," *Asian J. Phys. Chem. Sci.*, vol. 13, no. 3, pp. 1–38, 2025. https://doi.org/10.9734/ajopacs/2025/v13i3248

[40] T. Fujita, "Hypergraph and superhypergraph approaches in electronics: A hierarchical framework for modeling

power-grid hypernetworks and superhypernetworks," *J. Energy Res. Rev.*, vol. 17, no. 6, pp. 102–136, 2025. https://doi.org/10.9734/jenrr/2025/v17i6425

[41] B. V. S. Marcos, M. F. Willner, B. V. C. Rosa, F. F. R. M. Yissel, E. R. Roberto, L. D. B. Puma, and D. M. M. Fernández, "Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies," *Neutrosophic Sets Syst.*, vol. 84, pp. 513–524, 2025. https://doi.org/10.5281/zenodo.15641273

[42] Z. Chen, X. Wang, C. Wang, and J. Li, "Explainable link prediction in knowledge hypergraphs," in *Proceedings of the 31st ACM International Conference on Information & Knowledge Management*, 2022, pp. 262–271. https://doi.org/10.1145/3511808.355731

[43] A. Hogan, E. Blomqvist, M. Cochez, C. d'Amato, G. D. Melo, C. Gutierrez, S. Kirrane, J. E. L. Gayo, R. Navigli, S. Neumaier *et al.*, "Knowledge graphs," *ACM Comput. Surv.*, vol. 54, no. 4, pp. 1–37, 2021. https://doi.org/10.1145/3447772

[44] S. Ji, S. Pan, E. Cambria, P. Marttinen, and S. Y. Philip, "A survey on knowledge graphs: Representation, acquisition, and applications," *IEEE Trans. Neural Networks Learn. Syst.*, vol. 33, no. 2, pp. 494–514, 2021. https://doi.org/10.1109/TNNLS.2021.3070843

[45] M. Nickel, K. Murphy, V. Tresp, and E. Gabrilovich, "A review of relational machine learning for knowledge graphs," *Proc. IEEE*, vol. 104, no. 1, pp. 11–33, 2015. https://doi.org/10.1109/JPROC.2015.2483592

[46] B. N. Walker and M. A. Nees, "An agenda for research and development of multimodal graphs," in *Proceedings of the International Conference on Auditory Display (ICAD2005)*, 2005, pp. 428–432.

[47] W. Yu and S. Brewster, "Evaluation of multimodal graphs for blind people," *Univers. Access Inf. Soc.*, vol. 2, pp. 105–124, 2003. https://doi.org/10.1007/s10209-002-0042-6

[48] L. Zhang, J. Guo, J. Wang, J. Wang, S. Li, and C. Zhang, "Hypergraph and uncertain hypergraph representation learning theory and methods," *Mathematics*, vol. 10, no. 11, p. 1921, 2022. https://doi.org/10.3390/math10111921

[49] H. W. Hamber, "Quantum gravity on the lattice," *Gen. Relativ. Gravitation*, vol. 41, pp. 817–876, 2009. https://doi.org/10.1007/s10714-009-0769-y

[50] R. A. Borzooei, Y. B. Jun, M. M. Takallo, S. Khademan, M. Sabetkish, and A. Pourderakhshan, "Double-framed soft filters in lattice implication algebras," *Ann. Univ. Craiova Math. Comput. Sci. Ser.*, vol. 47, no. 2, pp. 294–307, 2020. https://doi.org/10.52846/ami.v47i2.1181

[51] M. Li, X. Shi, C. Qiao, T. Zhang, and H. Jin, "Hyperbolic hypergraph neural networks for multi-relational knowledge hypergraph representation," *arXiv preprint arXiv:2412.12158*, 2024. https://doi.org/10.48550/arXiv.2412.12158

[52] S. Bermudo, J. M. Rodríguez, J. M. Sigarreta, and J. M. Vilaire, "Gromov hyperbolic graphs," *Discrete Math.*, vol. 313, no. 15, pp. 1575–1585, 2013. https://doi.org/10.1016/j.disc.2013.04.009

[53] E. Jonckheere, P. Lohsoonthorn, and F. Bonahon, "Scaled gromov hyperbolic graphs," *J. Graph Theory*, vol. 57, no. 2, pp. 157–180, 2008. https://doi.org/10.1002/jgt.20275

[54] L. Gugelmann, K. Panagiotou, and U. Peter, "Random hyperbolic graphs: Degree sequence and clustering," in *Automata, Languages, and Programming: 39th International Colloquium, ICALP 2012,*. Springer Berlin Heidelberg, 2012, pp. 573–585. https://doi.org/10.1007/978-3-642-31585-5_51