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Hexagonal Fuzzy-Based Review on Imperfect EPQ Models Involving Rework and Lost Sales Penalties

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Abstract: Imperfect production and rework in contemporary manufacturing systems, are inevitable realities hampering overall performance and cost efficiency. To address this challenge, this study developed an Economic Production Quantity (EPQ) model which integrated defective items, rework, disposal, and penalties for lost sales within a fuzzy decision-making framework. The convexity of the model implied the possible existence of an optimal solution. Compared to conventional crisp models, the proposed approach provided a more robust and realistic evaluation of inventory and cost structures by representing indeterminate parameters such as production cost, backordering cost, and penalty cost through Hexagonal Fuzzy Numbers (HFNs) and Graded Mean Deviation Method (GMDM) for defuzzification. The numerical illustration demonstrated superiority of the fuzzy model in minimizing the total cost, balancing inventory levels, and enhancing service quality. Sensitivity analysis further highlighted the adaptability of the model to combat unpredictable changes in the parameters. The study concluded with valuable insights for decision-makers to optimize imperfect production processes, strengthen resource allocation, and tackle uncertainty in real-world manufacturing environment.

Keywords: Imperfect production; Economic Production Quantity (EPQ); Hexagonal Fuzzy Numbers (HFNs); Rework; Lost sales penalty; Fuzzy inventory optimization

1 Introduction

This research discusses imperfect quality reworkable products with penalty lost sale and Hexagonal Fuzzy Numbers (HFN). Production cost, backordered price and penalty bygone sale price could be treated like fuzzy parameters. A Graded Mean Difference Method (GMDM) is applied for the method of defuzzification. While several prior studies have applied fuzzy set approaches in manufacturing systems, this research introduces a novel concept—the use of HFN in modelling the rework process.

Taleizadeh et al. [1] explained an EPQ model where manufacturing imperfect rate sticks to a uniform distribution. Finally, they prove that, they obtained derivation is to be convex and the entire price is minimized. Singh et al. [2] updated a sample to supply chain network with reverse logistics inventory sample providing a remanufacturing of returned items. The solution for this method is solved by hessian matrix to obtain the minimum total average cost. Gani and Sabarinathan [3] investigates an optimal ordering strategy for inventory systems under a fuzzy framework. The main focus is on maximizing the retailer's profit within a single production cycle, where cost parameters are modeled as fuzzy numbers to address uncertainty. Goyal et al. [4] presented the paper with demand variation based on the time. They made use of genetic algorithm to derive the EPQ sample for reducing the entire relevant amount. Singh et al. [2] designed a production sample using ramp type order rate with trade credit period is allowed in two categories.

Shastri et al. [5] examined the EOQ with two categories of business offers for analyzing how to manage the chain of supplies. A retailer's plays a major role in this model. Three algorithms are discussed to get an optimal replenishment cycle time. Tayal et al. [6] discussed the paper with seasonal products which undergo weibull deterioration with two markets namely primary and alternative. Priyan et al. [7] explained with an EOQ inventory sample which has two kinds of fuzzy numbers namely triangular and trapezoidal. Classical differential optimization technique is adopted for finding an optimal solution. Priyan and Uthayakumar [8] designed the degradation rate in TFN and startup investment is decreased by investing extra amount. Finally, it was shown that the total cost of the production is minimized. Shah and Cardenas-Barron [9] designed an inventory sample for analyzing the concepts of offering the trade credit to the customers and retailers. For retailers, the cash discount and the order link trade credits are allowed.

Kumar et al. [10] analyzed the design of an inventory sample which considered decaying range as weibull. In this paper the optimal cycle lengths as well as output for the profitable maintenance under different cost considerations are derived logically. Singh et al. [11] framed the paper with inventory reliability demand and demand rate based on seasons. This paper aims for optimizing the entire amount of the production process and finding the total quantity. Tayal et al. [12] discussed the manufacturing inventory distribution sample for decaying items. This paper aims to reduce the net amount. The supply chain concept is developed for the suppliers and the buyers.

Sahoo and Dash [13] developed a model addressing fuzzy probabilistic samples, considering fuzzy demand and fuzzy storage capacity for a single period. Their findings revealed that total profit could be maximized in both overstocking and understocking situations. Raula et al. [14] examined a manufacturing model under both crisp and fuzzy environments, where all parameters were expressed as hexagonal fuzzy numbers. Sahoo et al. [15] analyzed a system where holding cost and demand rate were modeled as pentagonal fuzzy numbers, applying the graded mean difference method for defuzzification of total cost.

Jauhari et al. [16] investigated an EPQ inventory model with defective production and fuzzy annual demand, emphasizing shipment optimization between seller and customer. San-Jose et al. [17] introduced an inventory framework dependent on price and time, proposing a novel algorithm to enhance profit per unit time. Agi and Soni [18] formulated a deterministic model in which demand was influenced by product quality and freshness. Kirchi et al. [19] explored upstream and production strategies for both retailers and manufacturers to minimize large-scale production costs.

Alfares and Ghaithan [20] provided a survey of EOQ and EPQ models under varying holding cost structures. Pakhira et al. [21] considered an EOQ model incorporating memory effects in a fuzzy environment, where defuzzification was carried out using the graded integration approach. Mahapatra et al. [22] proposed a fuzzy inventory model with trade credit and permissible delay in payment, employing symmetric triangular fuzzy numbers for fuzzification and λ-integral for defuzzification. Kuppulakshmi et al. [23] extended an EPQ framework to estimate annual fish production and single-period supply to retailers through an efficient allocation process. Dong et al. [24] designed a production model incorporating multiple pricing strategies to achieve the minimum total cost. Vadivel et al. [25] presented an EPQ-style production planning model extended to a single-stage system with rework, explicitly incorporating hexagonal fuzzy numbers to model uncertainty and showing cost reductions under imperfect production.

Kumar et al. [26] compared two EPQ variants (with and without shortages) that include rework and scrap; the paper uses fuzzy modelling and references hexagonal fuzzy approaches in its methodology. The conference PDF contains the model formulations and numerical examples. Yiğit [27] developed hexagonal type-2 fuzzy set methods for inventory decisions (safety stock). While this paper focuses on safety-stock and HT2FS, its methods and HFN extensions are highly relevant for EPQ models that require robust ranking/defuzzification under ambiguity. Hemalatha and Annadurai [28] compared triangular, trapezoidal, pentagonal and hexagonal fuzzy numbers for inventory optimization (EOQ); useful for methodology and defuzzification comparison when applying HFNs to EPQ/rework problems. Arora et al. [29] used hexagonal fuzzy numbers to fuzzify cost parameters in a remanufacturing/production model (includes remanufacturing/rework aspects) and applies graded mean integration for defuzzification. Shows HFN use for remanufacturing + environmental constraints. Arora et al. [30] developed an EPQ-type optimization that explicitly treats imperfect quality, rework and repair; although several fuzzy representations are used across the literature, this paper ties imperfect products + rework into production planning and is useful background for direct HFN extensions.

The fundamental definitions are adopted from Zadeh (1965).

Definition 1.1

A hexagonal fuzzy number $\tilde{\Delta} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is represented with membership function $\mu_{\breve{A}}$.

$$\mu_{\breve{A}} = \begin{cases} L_1(\mathfrak{X}) = \frac{1}{2}\left(\frac{\mathfrak{X}-m_1}{m_2-m_1}\right), & m_1 \leq \mathfrak{X} \leq m_2 \\ L_2(\mathfrak{X}) = \frac{1}{2} + \frac{1}{2}\left(\frac{\mathfrak{X}-m_2}{m_3-m_2}\right), & m_2 \leq \mathfrak{X} \leq m_3 \\ 1, & m_3 \leq \mathfrak{X} \leq m_4 \\ R_1(\mathfrak{X}) = 1 - \frac{1}{2}\left(\frac{\mathfrak{X}-m_4}{m_5-m_4}\right), & m_4 \leq \mathfrak{X} \leq m_5 \\ R_2(\mathfrak{X}) = \frac{1}{2}\left(\frac{m_6-\mathfrak{X}}{m_6-m_5}\right), & m_5 \leq \mathfrak{X} \leq m_6 \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{\Delta} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is the $\propto$ cut, $0 \leq\propto\leq 1$ is $\Delta(\propto) = [\Delta_L(\propto), \Delta_R(\propto)]$

$$\Delta_{L_1}(\propto) = m_1 + (m_2 - m_1)\propto = L_1^{-1}(\propto)$$
$$\Delta_{L_2}(\propto) = m_2 + (m_3 - m_2)\propto = L_2^{-1}(\propto)$$
$$\Delta_{R_1}(\propto) = m_3 + (m_3 - m_4)\propto = R_1^{-1}(\propto)$$
$$\Delta_{R_2}(\propto) = m_6 + (m_6 - m_5)\propto = R_2^{-1}(\propto)$$

Hence,

$$L^{-1}(\propto) = \frac{L_1^{-1}(\propto) + L_2^{-1}(\propto)}{2} = \frac{m_1 + m_2 + (m_3 - m_1)\propto}{2}$$
$$R^{-1}(\propto) = \frac{R_1^{-1}(\propto) + R_2^{-1}(\propto)}{2} = \frac{m_5 + m_6 + (m_4 - m_6)\propto}{2}$$

Definition 1.2

$\tilde{\Delta} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is HFN then the graded mean representation (GMDM) of $\tilde{\Delta}$ is defined as

$$\mathrm{P}(\tilde{\Delta}) = \frac{1}{12}\left[m_1 + 3m_2 + 2m_3 + 2m_4 + 3m_5 + m_6\right]$$

Definition 1.3

Suppose $\tilde{A} = (m_1, m_2, m_3, m_4, m_5, m_6)$ and $\tilde{B} = (n_1, n_2, n_3, n_4, n_5, n_6)$ are two HFN's m_1, m_2, m_3, m_4, m_5, m_6, n_1, n_2, n_3, n_4, n_5, n_6 are all real numbers.

1. $\tilde{A} + \tilde{B} = (m_1 + n_1, m_2 + n_2, m_3 + m_3, m_4 + n_4, m_5 + n_5, m_6 + n_6)$.
2. $\tilde{A} * \tilde{B} = (m_1 n_1, m_2 n_2, m_3 n_3, m_4 n_4, m_5 n_5, m_6 n_6)$.
3. $-\tilde{B} = (-n_6, -n_5, -n_4, -n_3, -n_2, -n_1)$.
4. $\frac{1}{\tilde{B}} = \tilde{B}^{-1} = \left(\frac{1}{n_6}, \frac{1}{n_5}, \frac{1}{n_4}, \frac{1}{n_3}, \frac{1}{n_2}, \frac{1}{n_1}\right)$.
5. $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{m_1}{m_6}, \frac{m_2}{m_5}, \frac{m_3}{m_4}, \frac{m_4}{m_3}, \frac{m_5}{m_2}, \frac{m_6}{m_1}\right)$.
6. Let $\propto \in R$, then

$$\alpha * \tilde{A} = \begin{cases} (\alpha m_1, \alpha m_2, \alpha m_3, \alpha m_4, \alpha m_5, \alpha m_6), & \alpha \geq 0 \\ (\alpha n_1, \alpha n_2, \alpha n_3, \alpha n_4, \alpha n_5, \alpha n_6), & \alpha < 0 \end{cases}$$

2 Basic Assumption

• A single machine operating with multiple setups is considered.
• The planning horizon is finite.
• Defective items generated during production can be reworked.
• Backlogging is permitted, and unmet demand is treated as a penalty for lost sales.
• Some defective units may be discarded as scrap during the rework process.
• In a fuzzy environment, factors such as production cost, backlogging cost, and penalty for lost sales are taken into account.

3 Notations

A - Expected setup cost per cycle
Q - Production Quantity per cycle
θ - Defective production rate
P - Normal production rate
P' - Rework rate of defective item
D - Demand rate (where $P > DP > D$)
X - Proportion of defective items
f - Fraction of demand backordered
I^{max} - Maximum inventory after rework
R_c - Rework Cost per defective items
D_c - Disposal cost of scrabbed items
H_c - Holding cost for reworked items
${H_c}'$ - Anticipated total holding cost for rework able items
B_c - shortage cost
P_c - Production cost
L_c - Lost sales cost
I - Inventory level during production
B - Backordered quantity

$TC(Q,B)$ - Total cost of production process
$E(.)$ - Expected value operator
θ' - Rework production rate
γ - Number of scrapped items during rework
τ_1 - Production time
τ_2 - Idle time (Non production time)
τ_3 - Shortage period
τ_4 - Time to clear backorders
τ_5 - Rework time

4 Mathematical Model

The production model is adapted from Taleizadeh et al. [1], which considers a single machine with limited capacity operating under multiple production processes. In the present work, this framework is extended by incorporating fuzzy concepts. The objective is to determine the total cycle time and backordered quantity more effectively by introducing hexagonal fuzzy numbers as parameters. Production commences at $\tau = 0$, and throughout the process, the production rate (P) is assumed to be greater than or equal to the demand rate (D). The proportion of imperfect items (θ) is illustrated in Figure 1.

$$P - \theta - D \geq 0, \quad 0 \leq X \leq \left(1 - \frac{D}{P}\right) \tag{1}$$

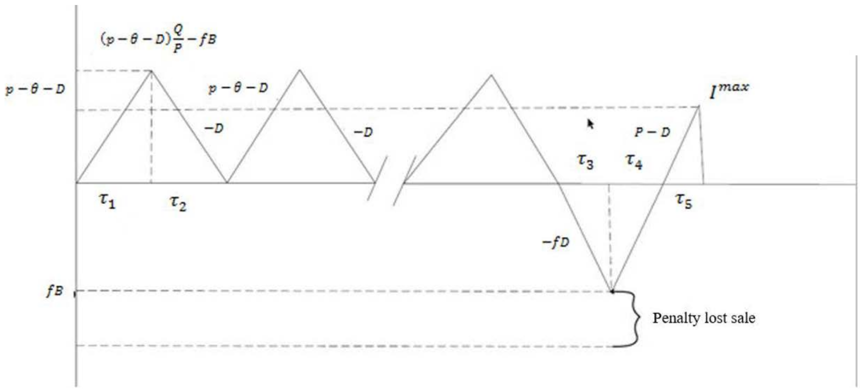

Figure 1. EPQ inventory model with n - cycle

The duration of manufacturing cycle length is the union of production, non production, backordered permitted time and rework time. The total cycle time during the production process is given as

$$T = \sum_{i=1}^{5} \tau_i \tag{2}$$

The total inventory level during the manufacturing process is defined as

$$I = (p - \theta - D)\frac{Q}{P} - fB \tag{3}$$

During the time τ_1 the production process takes place and some products are damaged, the demand also satisfied during the same cycle time. Because of deterioration of some products some goods are disposed as a scrap.

$$\tau_1 = \frac{I}{(p - \theta - D)} \tag{4}$$

The production stops after the time period τ_1 and the demand takes place during the time τ_2:

$$\tau_2 = \frac{(p - \theta - D)\frac{Q}{P} - fB}{D} \tag{5}$$

Due to the demand of the goods, the products are back ordered and this takes place during the time τ_3 and τ_4:

$$\tau_3 = \frac{B}{D} \tag{6}$$

$$\tau_4 = \frac{fB}{(p - \theta - D)} \tag{7}$$

All the rework able items are collected separately after the n cycles, then it will be reworked separately during the period (τ_5). Waste items during the rework process are separated as scarp.

$$\tau_5 = \frac{XQ}{P'} = \frac{\theta Q}{PP'} \tag{8}$$

The total items which is produced during the rework period is given as

$$I^{max} = n\left(P' - \theta' - D\right)\tau_5 \tag{9}$$

The scraped items rate during rework is given by

$$\theta' = P'\gamma \quad \text{with} \quad 0 \le \gamma \le 1 \tag{10}$$

$$\theta'\tau_5 = \frac{\gamma Q\theta}{P} \tag{11}$$

During the manufacturing time, γ- rate of scrabbed items are produced

$$\theta = PX \tag{12}$$

Eq. (11) can be written as

$$\theta'\tau_5 = X\gamma Q \tag{13}$$

The total quantity of one production is given by

$$Q = \frac{DT - (1-f)B}{(1-\gamma X)}, \quad \text{where} \quad 0 \le \gamma \le 1 \tag{14}$$

The total cost for n-production setup and one rework setup is given by

$$\begin{aligned} TC(Q,B) = {} & \textit{Production cost} + \textit{Mending price for deterioration cost} + \textit{Disposal cost} \\ & + \textit{Holding cost for non defective items} + \textit{Rework cost} \\ & + \textit{Holding cost for reworked items} + \textit{Back ordered cost} + \textit{Lost sale cost} \\ & + \textit{Aniticipated total setup cost} \end{aligned}$$

$$\begin{aligned} TC(Q,B) = {} & \frac{P_cQ}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] + \\ & \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_cfB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_c}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T} \end{aligned} \tag{15}$$

Fuzzy model

In the crisp model, all parameters are considered fixed and determined with certainty. However, in practice, obtaining exact values is often unrealistic. To address this, the problem is examined under a fuzzy environment, where manufacturing cost, penalty for lost sales, and backordering cost are represented as fuzzy numbers.

$\tilde{P_c}$ = fuzzy production cost

$\tilde{B_c}$ = fuzzy backordered cost

$\tilde{L_c}$ = fuzzy lost sale cost

Suppose $\tilde{P_c} = (P_1, P_2, P_3, P_4, P_5, P_6)$, $\tilde{B_c} = (B_1, B_2, B_3, B_4, B_5, B_6)$, $\tilde{L_c} = (L_1, L_2, L_3, L_4, L_5, L_6)$ are non negative hexagonal fuzzy numbers (HFN).

The total production cost is given below:

$$\widetilde{TC}(Q,B) = \left(\tilde{TC}_1(Q,B), \tilde{TC}_2(Q,B), \tilde{TC}_3(Q,B), \tilde{TC}_4(Q,B), \tilde{TC}_5(Q,B), \tilde{TC}_6(Q,B)\right) \tag{16}$$

where,

$$\begin{aligned} \widetilde{TC}_1(Q,B) = {} & \frac{P_1Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\ & + \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_1fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_1}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T} \end{aligned} \tag{17}$$

$$
\begin{aligned}
\widetilde{TC}_2(Q,B) =& \frac{P_2Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\
&+ \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_2fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_2}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\widetilde{TC}_3(Q,B) =& \frac{P_3Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\
&+ \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_3fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_3}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\widetilde{TC}_4(Q,B) =& \frac{P_4Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\
&+ \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_4fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_4}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\widetilde{TC}_5(Q,B) =& \frac{P_5Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\
&+ \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_5fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_5}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\widetilde{TC}_6(Q,B) =& \frac{P_6Q}{T} + \frac{R_cXQ}{T} + \frac{D_cX\gamma Q}{T} + \frac{H_c}{T}\left[n\left(\frac{I}{2}\tau_1 + \frac{I}{2}\tau_2\right) + \frac{\theta\left(n\tau_1+\tau_4\right)\left(n\tau_1+\tau_4\right)}{2}\right] \\
&+ \frac{H_c'}{T}\left[n\left(\frac{P'\tau_5}{2}\tau_5\right)\right] + \frac{B_6fB}{2T}\left(\tau_3+\tau_4\right) + \frac{L_6}{T}\frac{(1-f)B}{2}\left(\tau_3+\tau_4\right) + \frac{A}{T}
\end{aligned} \tag{22}
$$

The defuzzification for hexagonal fuzzy number is defined by the graded mean difference formula is given as

$$
\begin{aligned}
\widetilde{TC}(Q,B) =& \frac{1}{12}\Big(\widetilde{TC}_1(Q,B) + 3\widetilde{TC}_2(Q,B) + 2\widetilde{TC}_3(Q,B) + 2\widetilde{TC}_4(Q,B) + 3\widetilde{TC}_5(Q,B) + \\
&\widetilde{TC}_6(Q,B)\Big)
\end{aligned} \tag{23}
$$

$$
\tilde{B} = \frac{1}{12}\left(\tilde{B}_1 + 3\tilde{B}_2 + 2\tilde{B}_3 + 2\tilde{B}_4 + 3\tilde{B}_5 + \tilde{B}_6\right) \tag{24}
$$

5 Numerical Example

The theoretical findings are validated through a numerical example, with certain parameter values adapted from the study [5]. The adopted values are: P = 4000/cycle, D = 600/cycle, P' = 800/cycle, A = 500/cycle, f = 0.3/cycle, P_c = \$15/unit, H_c = \$5/unit, $H_c{}'$ = \$2/unit, D_c = \$3/unit, B_c = \$10/unit, L_c = \$12/unit, R_c = 1. The parameters θ, γ are assumed to follow a uniform distribution, and their respective values are obtained for both perfect and defective items.

The use of a uniform distribution implies that the manufacturing rate of defective products has an equal probability of occurring across the specified range.

Table 1. Manufacturing rate of imperfect products (θ)

$X \sim U(a,b)$ a	b	$E(X)$	$\theta = PE(X)$
0	0.04	0.02	80
0	0.08	0.04	160
0	0.12	0.06	240
0	0.16	0.08	320
0	0.20	0.10	400

For different values of a and b, the production rate of imperfect items and the manufacturing rate of reworking items are calculated and they are shown in Table 1 and Table 2. Taking the first value of the Table 1 and Table 2 and substitute the remaining numerical values in Eq. (15). The total time, back order level, optimum quantity, total cost for the production process , inventory level for the production and rework inventory level are given as

$T = 0.2590$, $B = 413.68$, $Q = 1782$, $TC(Q,B) = 4822600$, $I = 1604$, $I^{max} = 89.83$

The production rate of reworked items obtained by uniform distribution to get the accurate values of θ'.

Table 2. Production rate of reworked items ($\theta\prime$)

$\gamma \sim U(a,b)$ a	b	$E(\gamma)$	$\theta' = P'E(X)$
0	0.02	0.01	8
0	0.06	0.03	24
0	0.10	0.05	40
0	0.14	0.07	56
0	0.18	0.09	72

Fuzzy model

Let the hexagonal fuzzy numbers be defined as $\tilde{P_c} = (5\ \ 10\ \ 15\ \ 15\ \ 20\ \ 25)$, $\tilde{B_c} = (6\ \ 8\ \ 10\ \ 10\ \ 12\ \ 14)$, $\tilde{L_c} = (8\ \ 10\ \ 12\ \ 12\ \ 14\ \ 16)$. Using these parameters, the fuzzy estimates for the system are obtained as: fuzzy total time $\tilde{T}^* = 0.2589$, fuzzy backorder level $\tilde{B}^* = 412$, fuzzy production quantity $\tilde{Q}^* = 1780$, fuzzy total cost $\widetilde{TC}(Q,B) = 4823600$, fuzzy inventory level and maximum fuzzy inventory $\tilde{I}^* = 1602$, $\tilde{I}^{max*} = 89$.

When comparing the crisp and fuzzy models, optimal solutions are derived for cycle time, backorder quantity, and inventory levels during both production and rework. The fuzzy results are defuzzified using the Graded Mean Difference Method (GMDM), which demonstrates the superiority of the fuzzy approach over the crisp case.

6 Sensitivity Analysis

From Table 3, it is observed that the setup cost (A) rises linearly as the parameter varies between –15% and +15%, which in turn causes the total cost to increase. The demand rate also shows an upward trend within the same range. In addition, the shortage fraction grows linearly from –15% to +15%, and as illustrated in Figure 2, this results in a reduction of the total cost.

Table 3. Parameter change in total cost

%	-15	-10	-5	5	10	15
A	4495600	4604600	4604600	4914600	5012900	5109000
D	3538000	3908300	4327200	5400000	6142000	7153100
F	5343600	5035800	4891600	4762200	4716400	4665700

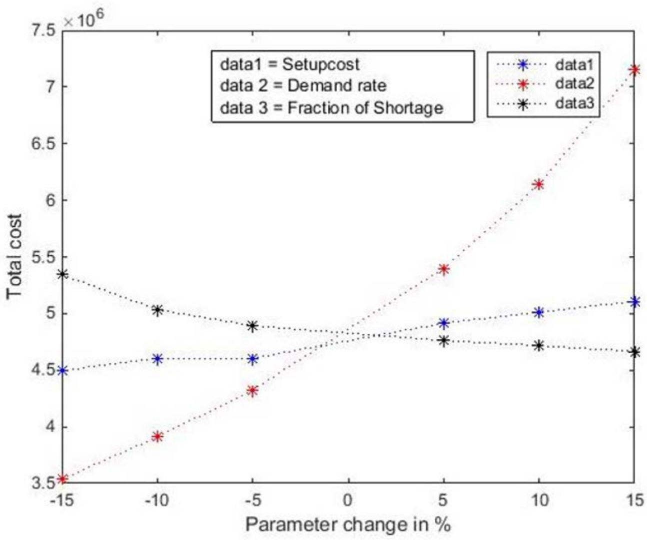

Figure 2. Variation of setup cost (A), demand rate (D), and shortage fraction (f) with parameter changes

Based on the results in Table 4, the production cost (P_c) increases linearly as the parameter varies from –15% to +15%, leading to a corresponding rise in total cost. The holding cost (H_c) also shows a linear increase over the same range; however, its impact on the total cost is twofold—between –15% and –5% the total cost decreases, while from +5% to +15% it rises linearly. Similarly, the backorder cost grows linearly across the –15% to +15% interval, which in turn elevates the total cost. As illustrated in Figure 3, the rework cost follows the same pattern, increasing linearly with parameter variation, thereby causing the total cost to increase accordingly.

Table 4. Parameter change in total cost

%	-15	-10	-5	5	10	15
P_c	4703600	4740300	4777100	4850600	4887400	4924100
H_c	6414200	4919300	4711500	5060000	5396000	5799100
B_c	4813300	4813500	4813700	4814100	4814200	4814400
R_c	4813700	4813800	4813800	4813900	4814000	4414000

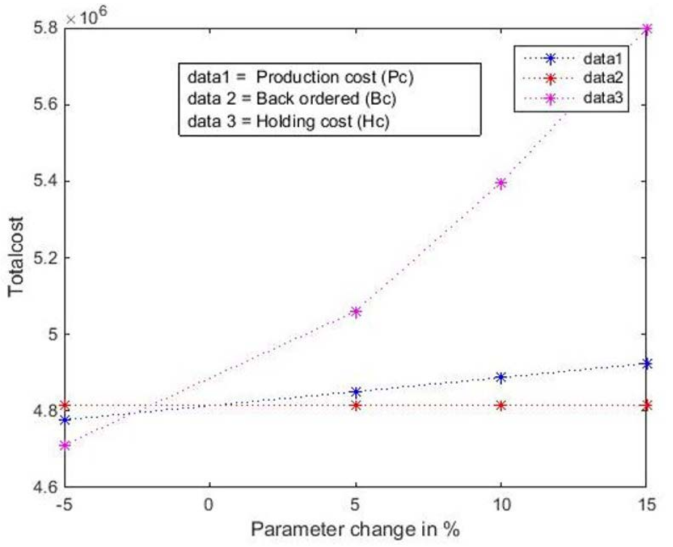

Figure 3. parameter change in Production cost (P_c), backordered cost (B_c) and holding cost (H_c)

7 Conclusion

This paper proposed an EPQ model for imperfect production systems incorporating rework, scrap, and lost sales penalties under both crisp and fuzzy environments. The adoption of Hexagonal Fuzzy Numbers and the Graded Mean Deviation Method for defuzzification provided greater flexibility in handling uncertainty, yielding more realistic and cost-effective results than conventional models. The fuzzy-based framework not only minimized total production costs but also improved inventory balance, even when defective items were partially scrapped. Numerical results and sensitivity analyses confirmed the robustness of the approach across varying system parameters. The contributions of this work lie in establishing a structured methodology for managing reworkable items under uncertainty and demonstrating its practical relevance for modern manufacturing. Future research directions include extending the model to multi-stage supply chains, integrating advanced fuzzy and neutrosophic environments, and exploring hybrid decision-making algorithms for large-scale industrial applications.

Author Contributions

All authors contributed to the study conception and design. Conceptualization, formal analysis, investigation, data curation, writing-original draft: K.V. and S.C.; resources and supervision: N.D.; methodology: K.V., S.C. and N.D.; writing-review & editing, visualization: K.V., S.C. and N.D. All authors have read and agreed to the published version of the manuscript.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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