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Leveraging the Bipolar Fuzzy Numbers in Data Envelopment Analysis to Enhance the Performance Evaluation

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Abstract: Fuzzy data envelopment analysis (FDEA) plays an essential role in the current socio-economic scenario to analyze the performance of decision-making units (DMUs) within a fuzzy environment. This paper introduced a novel Bipolar Fuzzy Data Envelopment Analysis (BFDEA) model using bipolar triangular fuzzy numbers to accommodate both uncertainty and ambiguity in evaluating the performance of a finite number of DMUs. The BFDEA model utilizes a value function for bipolar fuzzy numbers and translates BFDEA models into equivalent crisp models, thus providing thorough and precise evaluations of efficiency. The BFDEA model embraces a super-efficiency framework to offer a full ranking of efficient DMUs, while establishing a benchmarking framework for a meaningful discussion of improvements in performance. A numerical example showed that the BFDEA method could provide a reliable nuanced evaluation even in the presence of conflicting information. This work contributes to the DEA literature, where uncertainty has been inadequately addressed up till the present, by providing breakthroughs in a convincing way for decision makers to analyze performance amidst complicated and indeterminate situations.

Keywords: Data envelopment analysis; Bipolar fuzzy number; Performance evaluation; Super-efficiency model; Benchmarking technique; Value function

1 Introduction

In the fields of economics, social sciences, and engineering, the challenges that are confronted frequently show underlying uncertainties, a lack of accuracy, and ambiguities. The confusing and inadequate information that are characteristic of actual conditions are the source of these issues. Zadeh [1] proposed fuzzy set theory to successfully solve these concerns. This innovative concept established a systematic framework for addressing imprecise data, allowing academics and practitioners to model and evaluate uncertainty more efficiently in various domains. Fuzzy set theory provides a paradigm for addressing uncertainty and ambiguity in intricate real-world systems. It offers a crucial method for capturing non-probabilistic characteristics of uncertain, incomplete, imprecise, or ambiguous information. Zhang [2] introduced YinYang bipolar fuzzy set (BFS), a dual-faceted logical framework for managing opposing elements of information. In a BFS, each element is characterized by two components: the first, which ranges from (0, 1], denotes the membership value for a certain property of the fuzzy set, while the second, which ranges from [–1, 0), indicates the membership value for the opposing property of the fuzzy set. In the last two decades, several scholars have investigated and utilized BFSs in diverse domains [3–5]. BFSs have garnered significant interest in recent years, leading to heightened study and applications [6]. Bipolar fuzzy graphs were initially introduced by Akram [7, 8], and subsequently, Akram et al. [9, 10] investigated decision support systems utilizing bipolar fuzzy graphs.

Data Envelopment Analysis (DEA) is a prevalent non-parametric methodology for assessing the relative efficiency of Decision-Making Units (DMUs) that utilize numerous inputs and outputs. The basic DEA model, established by Charnes et al. [11] and usually known as the CCR model, presumes constant returns to scale (CRS). This methodology assesses the efficiency of DMUs by analyzing their input-output ratios and determining a best-practice frontier. The CCR model is fundamental and broadly applicable; nonetheless, its intrinsic assumptions and limitations have resulted in many modifications and changes over time. Banker et al. [12] proposed the BCC model, which modifies the constant returns to scale (CRS) assumption to include variable returns to scale (VRS). Subsequently, other researcher have expanded the DEA model to develop various alternative DEA models, each offering a distinct perspective. The extensions comprise the SBM model [13], the additive model [14], the undesirable model [15], the super-efficiency model [16], the network DEA model, the dynamic DEA model and so on. Each model offers a distinct methodology and enhances the overall growth and innovation of the DEA field. It has been widely utilized in several areas, such as agriculture, finance, education, healthcare, and transportation.

The transition from standard DEA to Fuzzy DEA (FDEA) is a significant progression in efficiency analysis, specifically in tackling the complexities associated with real-world data that is uncertain and imprecise. Traditional DEA models assume the existence of exact, or "crisp", input and output data. In several real situations, data might be confusing, partial, or inaccurate, making such exact models less useful or even irrelevant. To address these constraints, FDEA incorporates fuzzy set theory into DEA, offering a comprehensive framework for managing and analyzing DMUs in uncertain settings. This methodological transition allows for the integration of fuzzy numbers, intervals, and linguistic factors, so providing a more accurate depiction of confusing data. The shift to FDEA has prompted a range of methodological advancements and enhancements, encompassing the creation of fuzzy efficiency metrics, fuzzy frontiers, and ranking techniques for DMUs functioning in uncertain contexts. Sengupta [17], in a pioneering effort, was the first to utilize fuzzy sets in the context of performance evaluation problems to gauge the relative efficiency of DMUs in an uncertain environment. This groundbreaking work served as an inspiration for a multitude of authors who subsequently developed different solution approaches, which can be categorized as follows: (i) the possibility approach, (ii) the ranking approach, (iii) the α-cut approach, (iv) the tolerance approach, (iv) the type-2 fuzzy set approach, (v) the arithmetic approach, (vi) the multi-objective approach, and so on [18, 19]. FDEA has garnered substantial interest in assessing performance among organizations in many different sectors, particularly under conditions of uncertainty. Its use has been particularly noted in sectors for Agriculture, Healthcare, Education, Energy, and resource allocation models [20–24]. FDEA has seen large advancements through the incorporation of several other extensions and generalization of fuzzy sets [25] such as Intuitionistic FDEA [26, 27], Pythogorean FDEA [28, 29], Neutrosophic DEA [26, 30–33] and Spherical FDEA [34, 35]. These extensions consider complex cases in decision making involving imprecision and uncertainty.

Motivation for development of efficiency measurement and complete ranking technique in bipolar fuzzy environment is discuss here. The rationale for incorporating a BFS into the framework of DEA stems from the recognition of the limitations inherent in traditional DEA models when confronted with uncertainties and imprecise information. In numerous real-world scenarios, DMUs operate within environments that are characterized by ambiguity and vagueness, thereby presenting a formidable challenge in accurately assessing their efficiency by means of conventional approaches. The introduction of a BFS in DEA is underpinned by the necessity to capture both positive and negative aspects of uncertainty, thereby acknowledging the fact that DMUs may not only exhibit excellence but also confront inefficiencies or suboptimal performance. BFS provide a more expressive and flexible representation of uncertainty in contrast to standard fuzzy sets. By permitting membership degrees to span across both positive and negative values, they facilitate a nuanced modeling of imprecision, thus rendering them highly suitable for situations in which decision-makers are required to take into consideration not only positive indicators of efficiency but also potential inefficiencies or deviations from the ideal performance. The incorporation of BFS into DEA serves to enhance the model's capacity to handle conflicting and contradictory information, thereby providing decision-makers with a more comprehensive and realistic assessment of DMUs' efficiency within uncertain environments. This particular approach proves to be particularly invaluable in sectors where decision-making entails inherent vagueness or when performance evaluations necessitate the consideration of both positive and negative deviations from the optimal efficiency frontier. Ultimately, the integration of BFS into DEA contributes to the establishment of a more robust and reliable framework for the evaluation and enhancement of the efficiency of DMUs within complex and uncertain settings.

The main contribution of this manuscript is highlighted as follows.

1. The valued function is proposed for bipolar triangular fuzzy number to order the BTFNs.

2. To measure the relative efficiency of the DMUs in bipolar fuzzy environment, we have proposed Bipolar fuzzy data envelopment analysis (BFDEA).

3. In order to completely ranked the efficient DMUs in bipolar fuzzy environment, bipolar fuzzy super-efficiency (BFSE) model is proposed.

4. A novel solution technique is developed for solving the BFDEA model and BFSE model to measure the realative efficiency and complete ranking of the DMUs.

The remaining sections of the paper are structured as follows: Section 2 introduces the basic definition and fundamental concept of BFS. A novel valued function is proposed for bipolar trapezoidal fuzzy number and bipolar triangular fuzzy number by using the α, β-cut of BFS. In section 3, we develop bipolar fuzzy DEA model by incorporating the inputs and outputs are bipolar triangular number in the traditional DEA model. In section 4 discuss the solution procedure of the proposed BFDEA model. Section 5 shows the existence and applicability of the proposed model through a numerical example. Section 6 concludes with advantage, limitation and future direction of the proposed approach.

2 Bipolar Fuzzy Set

This section discuss about basic definitions of fuzzy set, bipolar fuzzy set, trapezoidal bipolar fuzzy numbers and its arithmetic properties. Also a new ranking function is defined and deeply studies its properties.

Definition 1 (Fuzzy set) [1]. The fuzzy set (FS) $\widehat{A}$ in Ω is defined as

$$\widehat{A} = \{\langle x, \mu_A \rangle : x \in \Omega\}, \tag{1}$$

where, the function $\mu_A : \Omega \to [0,1]$ is the membership grade.

Definition 2 (Bipolar fuzzy set) [2]. The bipolar fuzzy set (BFS) $\widehat{B}$ in Ω is defined as

$$\widehat{B} = \{\langle x, \mu_B^+, \mu_B^- \rangle : x \in \Omega\}, \tag{2}$$

where, the functions $\mu_B^+ : \Omega \to [0,1]$ and $\mu_B^- : \Omega \to [-1,0]$ are the positive and negative membership grades.

Definition 3 ((α,β)-cut for BFS). Let $(\alpha,\beta) \in [0,1] \times [-1,0]$, then (α,β)-cut of the BFS $\widehat{B}$ are defined as $(\widehat{B}_\alpha, \widehat{B}_\beta)$ where,

$$\widehat{B}_\alpha = \left\{x \in \Omega : \mu_B^+(x) \geq \alpha\right\} \tag{3}$$

$$\widehat{B}_\beta = \left\{x \in \Omega : \mu_B^-(x) \leq \beta\right\} \tag{4}$$

Definition 4 [36]. A triangular bipolar fuzzy number (TBFN) is denoted by $\widehat{A} = \langle \overline{a}^L, \overline{a}^M, \overline{a}^U; \underline{a}^L, \underline{a}^M, \underline{a}^U \rangle$, and the positive and negative membership degrees as

$$\mu_A^+(x) = \begin{cases} f(x) = \frac{x-\overline{a}^L}{\overline{a}^M-\overline{a}^L}, & \overline{a}^L \leq x \leq \overline{a}^M \\ 1 & , x = \overline{a}^M \\ g(x) = \frac{\overline{a}^U-x}{\overline{a}^U-\overline{a}^M}, & \overline{a}^M \leq x \leq \overline{a}^U \\ 0 & , \text{Otherwise} \end{cases} \tag{5}$$

$$\mu_A^-(x) = \begin{cases} h(x) = \dfrac{\underline{a}^L - x}{\underline{a}^M - \underline{a}^L}, \underline{a}^L \leq x \leq \underline{a}^M \\ -1, x = \underline{a}^M \\ k(x) = \dfrac{x - \underline{a}^U}{\underline{a}^U - \underline{a}^M}, \underline{a}^M \leq x \leq \underline{a}^U \\ 0, \text{ Otherwise} \end{cases} \tag{6}$$

Graphical representation of TBFN is shown in Figure 1.

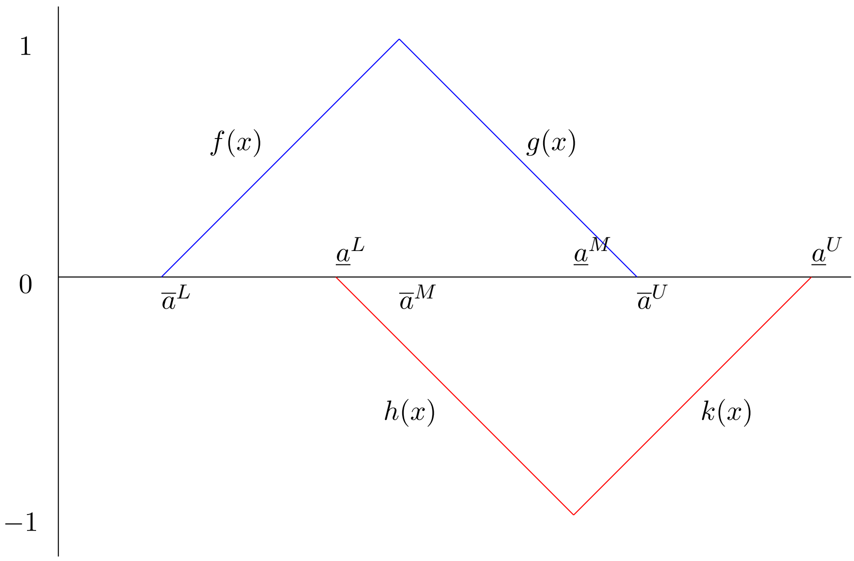

Figure 1. The positive and negative membership of TBFN

Definition 5 (Arithmetic properties). Let $\widehat{A}$ and $\widehat{B}$ are the two TrBFNs, denoted by $\widehat{A} = \langle \overline{a}^L, \overline{a}^M, \overline{a}^U; \underline{a}^L, \underline{a}^M, \underline{a}^U \rangle$ and $\widehat{B} = \langle \overline{b}^L, \overline{b}^M, \overline{b}^U; \underline{b}^L, \underline{b}^M, \underline{b}^U \rangle$. The arithmatic properties of the TrBFNs are defined as follows.

1. $\widehat{A} \oplus \widehat{B} = \langle \overline{a}^L + \overline{b}^L, \overline{a}^M + \overline{b}^M, \overline{a}^U + \overline{b}^U; \underline{a}^L + \underline{b}^L, \underline{a}^{M_1} + \underline{b}^{M_1}, \underline{a}^{M_2} + \underline{b}^{M_2}, \underline{a}^U + \underline{b}^U \rangle$
2. $\widehat{A} \ominus \widehat{B} = \langle \overline{a}^L - \overline{b}^U, \overline{a}^M - \overline{b}^M, \overline{a}^U - \overline{b}^L; \underline{a}^L - \underline{b}^U, \underline{a}^M - \underline{b}^M, \underline{a}^U - \underline{b}^L \rangle$
3. $\widehat{A} \otimes \widehat{B} = \langle \overline{a}^L \overline{b}^L, \overline{a}^M \overline{b}^M, \overline{a}^U \overline{b}^U; \underline{a}^L \underline{b}^L, \underline{a}^M \underline{b}^M, \underline{a}^U \underline{b}^U \rangle$
4. $\lambda \widehat{A} = \begin{cases} \langle \lambda \overline{a}^L, \lambda \overline{a}^M, \lambda \overline{a}^U; \lambda \underline{a}^L, \lambda \underline{a}^M, \lambda \underline{a}^U \rangle, & \textit{if } \lambda \geq 0. \\ \langle \lambda \overline{a}^U, \lambda \overline{a}^M, \lambda \overline{a}^L; \lambda \underline{a}^U, \lambda \underline{a}^M, \lambda \underline{a}^L \rangle, & \textit{if } \lambda \leq 0. \end{cases}$
5. $\dfrac{\widehat{A}}{\widehat{B}} = \langle \frac{\overline{a}^L}{\overline{b}^U}, \frac{\overline{a}^M}{\overline{b}^M}, \frac{\overline{a}^U}{\overline{a}^L}; \frac{\underline{a}^L}{\underline{a}^U}, \frac{\underline{a}^M}{\underline{b}^M}, \frac{\underline{a}^U}{\underline{b}^L} \rangle$

Definition 6 ((α, β)-cut for TBFN). Let $\widehat{A} = \langle \overline{a}^L, \overline{a}^M, \overline{a}^U; \underline{a}^L, \underline{a}^M, \underline{a}^U \rangle$ be a TBFN, then the (α, β)-cut for TBFN are defined as

$$\widehat{A}_\alpha = [L(\alpha), U(\alpha)] = \left[\overline{a}^L + \alpha(\overline{a}^M - \overline{a}^L),\ \overline{a}^U - \alpha(\overline{a}^U - \overline{a}^M)\right] \tag{7}$$

$$\widehat{A}_\beta = [L(\beta), U(\beta)] = \left[\underline{a}^M - \beta(\underline{a}^M - \underline{a}^L),\ \underline{a}^M + \beta(\underline{a}^U - \underline{a}^M)\right] \tag{8}$$

Definition 7 (Value index). The positive value index $V^+(\widehat{A})$ and negative value index $V^-(\widehat{A})$ with respect to the positive and negative membership of the a TBFN $\widehat{A}$ are defined as

$$V^+(\widehat{A}) = \int_0^1 (L(\alpha) + U(\alpha))\, f(\alpha) d\alpha \tag{9}$$

$$V^-(\widehat{A}) = \int_{-1}^0 (L(\beta) + U(\beta))\, g(\beta) d\beta \tag{10}$$

where, $f : [0, 1] \to [0, 1]$ is non-negative and increasing function such that $f(0) = 0, f(1) = 1$ and $\int_0^1 f(\alpha) d\alpha = 1/2$, and $g : [-1, 0] \to [0, 1]$ is non-negative and decreasing function such that $f(-1) = 0, f(0) = 1$ and $\int_{-1}^0 g(\beta) d\beta = 1/2$.

The functions $f(\alpha) = \alpha$ and $g(\beta) = -\beta$ are chosen in such a way that they satisfied the above mention conditions. The positive and negative value index for the BTFN are calculated as

$$V^+(\widehat{A}) = \frac{\overline{a}^L + 4\overline{a}^M + \overline{a}^U}{6} \tag{11}$$

$$V^-(\widehat{A}) = \frac{\underline{a}^L + \underline{a}^M + \underline{a}^U}{3} \tag{12}$$

Definition 8 (Value of TBFN). The value of TBFN $\widehat{A}$ is denoted as $V(\widehat{A})$ and is defined as

$$V(\widehat{A}) = \frac{V^+(\widehat{A}) + V^-(\widehat{A})}{2} \tag{13}$$

The value of TBFN is calculated as

$$V(\widehat{A}) = \frac{\overline{a}^L + 4\overline{a}^M + \overline{a}^U}{12} + \frac{\underline{a}^L + \underline{a}^M + \underline{a}^U}{6} \tag{14}$$

Definition 9 (Ordering of TBFN). Let $\widehat{A}$ and $\widehat{B}$ are the two TBFN.
1. If $\widehat{A} \leq \widehat{B}$ then $V(\widehat{A}) \leq V(\widehat{B})$
2. If $\widehat{A} \geq \widehat{B}$ then $V(\widehat{A}) \geq V(\widehat{B})$
3. If $\widehat{A} = \widehat{B}$ then $V(\widehat{A}) = V(\widehat{B})$

Example 2.1 Let $A = \langle 1, 5, 8; 3, 5, 6 \rangle$ and $B = \langle 2, 6, 8; 3, 4, 6 \rangle$ are two triangular bipolar fuzzy numbers.

Theorem 1. Let $\widehat{A_i} = \langle \overline{a_i}^L, \overline{a_i}^M, \overline{a_i}^U; \underline{a_i}^L, \underline{a_i}^M, \underline{a_i}^U \rangle, \quad$ for $i = 1, \ldots, n$ be the n TBFN in $\mathbb{R}$. Then

$$V\left(\sum_{i=1}^n \widehat{A_i}\right) = \sum_{i=1}^n V\left(\widehat{A_i}\right) \tag{15}$$

Corollary 1. Let $\widehat{A}$ and $\widehat{B}$ be the two TBFNs in $\mathbb{R}$ and $\lambda \in \mathbb{R}$ be a real number, then

$$V\left(\widehat{A} + \lambda \widehat{B}\right) = V\left(\widehat{A}\right) + \lambda V\left(\widehat{B}\right) \tag{16}$$

3 Bipolar Fuzzy Data Envelopment Analysis

Suppose that there are n DMUs each having m inputs and r outputs as represented by the vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^r$, respectively. We define the input matrix X as $X = [x_1, \cdots, x_m] \in \mathbb{R}^{m\times n}$, and the output matrix Y as $Y = [y_1, \cdots, y_r] \in \mathbb{R}^{r\times n},\ \ x_i \in \mathbb{R}^m,\ \forall\, i = 1, 2, \cdots, m,\ \ y_k \in \mathbb{R}^r,\ \forall\, k = 1, 2, 3, \cdots, r$ and assume that $X > 0$ and $Y > 0$. Charnes [11] developed the CCR model for measuring the efficiency of DMU_o i.e.

$$\begin{aligned} \max_{u,v}\ \theta_o &= \frac{\sum\limits_{k=1}^{r} u_k y_{ko}}{\sum\limits_{i=1}^{m} v_i x_{io}}, \\ \text{subject to } & \frac{\sum\limits_{k=1}^{r} u_k y_{kj}}{\sum\limits_{i=1}^{m} v_i x_{ij}} \le 1,\ j = 1, 2, \cdots n, \\ \text{and } & u_k \ge 0,\ \ k = 1, 2, \cdots, r, \\ & v_i \ge 0,\ \ i = 1, 2, \cdots, m. \end{aligned} \tag{CCR model}$$

The corresponding linear program (LP_o) is,

$$\begin{aligned} \max_{u,v}\ \theta_o &= \sum_{k=1}^{r} u_k y_{ko}, \\ \text{subject to } & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{k=1}^{r} u_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ \ j = 1, 2, \cdots n, \\ \text{and } & u_k \ge 0,\ \ k = 1, 2, \cdots, r, \\ & v_i \ge 0,\ \ i = 1, 2, \cdots, m. \end{aligned} \tag{17}$$

The Bipolar Fuzzy DEA model improves the conventional DEA methodology by using bipolar fuzzy numbers to represent inputs and outputs. Bipolar fuzzy numbers encompass both positive and negative assessments, facilitating a sophisticated examination of systems characterized by ambiguous or contradictory data. This approach assesses the efficiency of DMUs in uncertain conditions, providing a more adaptable and realistic framework for performance evaluation. The Bipolar Fuzzy DEA model is defined as follows.

$$\begin{aligned} \max_{u,v}\ & \sum_{k=1}^{r} u_k \widehat{y_{ko}}, \\ \text{subject to } & \sum_{i=1}^{m} v_i \widehat{x_{io}} = \widehat{1}, \\ & \sum_{k=1}^{r} u_k \widehat{y_{kj}} - \sum_{i=1}^{m} v_i \widehat{x_{ij}} \le \widehat{0},\ \ j = 1, 2, \cdots n, \\ \text{and } & u_k \ge 0,\ \ k = 1, 2, \cdots, r, \\ & v_i \ge 0,\ \ i = 1, 2, \cdots, m. \end{aligned} \tag{18}$$

where, the inputs $\widehat{x_{ij}}$ and outputs $\widehat{y_{kj}}$ are BTFNs.

3.1 Bipolar Fuzzy Super-Efficiency Model

The super-efficiency model is a sophisticated enhancement of the conventional DEA paradigm, designed to evaluate DMUs beyond the efficient frontier. In contrast to conventional DEA models that categorize DMUs as either efficient or inefficient, the super-efficiency model enables additional differentiation of efficient DMUs by excluding the assessed DMU from the reference set and recalculating its efficiency. This methodology offers a comprehensive ranking of DMUs, rendering it especially advantageous in situations when several DMUs are deemed efficient, hence addressing the problem of ties in efficiency rankings. It is widely utilized in performance assessment, resource distribution, and benchmarking across several domains. Significant contributions to the advancement of this model include [16], who initially introduced it as a method for conducting a more sophisticated efficiency investigation. The super-efficiency model is given as

$$\begin{aligned}
&\max_{u,v} \sum_{k=1}^{r} u_k y_{ko}, \\
&\text{subject to} \sum_{i=1}^{m} v_i x_{io} = 1, \\
&\sum_{k=1}^{r} u_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, 2, \cdots n; j \neq o \\
&\text{and } u_k \ge 0, \quad k = 1, 2, \cdots, r, \\
&v_i \ge 0, \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{super-efficiency model}$$

The bipolar fuzzy super efficiency (BFSE) model enhances the conventional super-efficiency framework by integrating bipolar fuzzy numbers to depict inputs and outputs, enabling a more sophisticated and accurate evaluation of DMUs functioning under uncertainty and ambiguity. This model defines inputs and outputs as bipolar fuzzy numbers, encapsulating both positive and negative dimensions of data, so representing the dual characteristics of rewards and opportunities alongside the hazards present in real-world situations. This method incorporates bipolar fuzzy numbers, so improving the modeling of imprecise and contradictory information and offering a more thorough ranking of DMUs, particularly in intricate contexts where conventional crisp or fuzzy models may be inadequate. The BFSE model is formally defined as follows:

$$\begin{aligned}
&\max_{u,v} \sum_{k=1}^{r} u_k \widehat{y_{ko}}, \\
&\text{subject to} \sum_{i=1}^{m} v_i \widehat{x_{io}} = \widehat{1}, \\
&\sum_{k=1}^{r} u_k \widehat{y_{kj}} - \sum_{i=1}^{m} v_i \widehat{x_{ij}} \le \widehat{0}, \quad j = 1, 2, \cdots n; j \neq o \\
&\text{and } u_k \ge 0, \quad k = 1, 2, \cdots, r, \\
&v_i \ge 0, \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{BFSE model}$$

4 Solution Procedure

The flow chart given in Figure 2, show that how to measure the efficiency score of the DMUs, how to completely ranked the DMUs and how to evaluate the peers for inefficient DMUs.

The following steps are required to solve the BFDEA model.

1. Construct the BFDEA model, given in Eq. (18) considering the inputs and outputs are in triangular bipolar fuzzy number.

2. The value function (V) is used to convert the BFDEA model into corresponding crisp LP model.

$$\begin{aligned}
&\max_{u,v} V\left(\sum_{k=1}^{r} u_k \widehat{y_{ko}}\right), \\
&\text{subject to } V\left(\sum_{i=1}^{m} v_i \widehat{x_{io}}\right) = V\left(\widehat{1}\right), \\
&V\left(\sum_{k=1}^{r} u_k \widehat{y_{kj}} - \sum_{i=1}^{m} v_i \widehat{x_{ij}}\right) \le V\left(\widehat{0}\right), \quad j = 1, 2, \cdots n, \\
&\text{and } u_k \ge 0, \quad k = 1, 2, \cdots, r, \\
&v_i \ge 0, \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{19}$$

From Definition 8 and Corollary 1, we have

$$
\begin{aligned}
\theta_o = \max_{u,v} & \sum_{k=1}^{r}\left(\frac{\overline{y_{ko}}^{L}+4\overline{y_{ko}}^{M}+\overline{y_{ko}}^{U}}{12}+\frac{\underline{y_{ko}}^{L}+\underline{y_{ko}}^{M}+\underline{y_{ko}}^{U}}{6}\right)u_k,\\
\text{s. t. } & \sum_{i=1}^{m}\left(\frac{\overline{x_{io}}^{L}+4\overline{x_{io}}^{M}+\overline{x_{io}}^{U}}{12}+\frac{\underline{x_{io}}^{L}+\underline{x_{io}}^{M}+\underline{x_{io}}^{U}}{6}\right)v_i = 1,\\
& \sum_{k=1}^{r}\left(\frac{\overline{y_{kj}}^{L}+4\overline{y_{kj}}^{M}+\overline{y_{kj}}^{U}}{12}+\frac{\underline{y_{kj}}^{L}+\underline{y_{kj}}^{M}+\underline{y_{kj}}^{U}}{6}\right)u_k\\
& \quad -\sum_{i=1}^{m}\left(\frac{\overline{x_{ij}}^{L}+4\overline{x_{ij}}^{M}+\overline{x_{ij}}^{U}}{12}+\frac{\underline{x_{ij}}^{L}+\underline{x_{ij}}^{M}+\underline{x_{ij}}^{U}}{6}\right)v_i \leq 0,\ j=1,2,\cdots n\\
\text{and } & u_k \geq 0,\ k=1,2,\cdots,r,\\
& v_i \geq 0,\ i=1,2,\cdots,m.
\end{aligned}
\tag{20}
$$

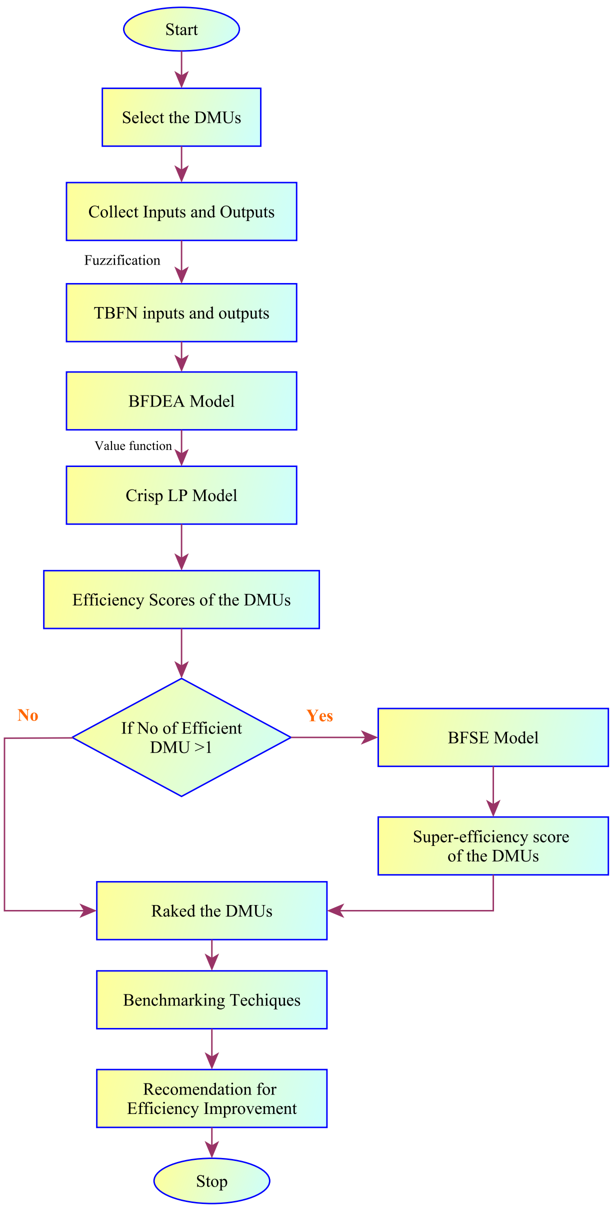

Figure 2. Technique for efficiency evaluation and peers calculation

3. Solve the above crisp LP model to find the optimal solution. The efficiency score of the DMU_o is θ_o^*, $(o = 1, 2, \cdots, n)$.

4. The DMUs are categorized into efficient and inefficient based on the given definition .

Definition 10 A DMU is said to be efficient if its efficiency score is one; otherwise it is considered as inefficient DMU.

5. The BFSE model converted into corresponding crisp LP model by using value function for TBFN.

$$
\begin{aligned}
\theta_o =& \max_{u,v} \sum_{k=1}^{r} \left(\frac{\overline{y_{ko}}^L + 4\overline{y_{ko}}^M + \overline{y_{ko}}^U}{12} + \frac{\underline{y_{ko}}^L + \underline{y_{ko}}^M + \underline{y_{ko}}^U}{6} \right) u_k, \\
\text{s. t. } & \sum_{i=1}^{m} \left(\frac{\overline{x_{io}}^L + 4\overline{x_{io}}^M + \overline{x_{io}}^U}{12} + \frac{\underline{x_{io}}^L + \underline{x_{io}}^M + \underline{x_{io}}^U}{6} \right) v_i = 1, \\
& \sum_{k=1}^{r} \left(\frac{\overline{y_{kj}}^L + 4\overline{y_{kj}}^M + \overline{y_{kj}}^U}{12} + \frac{\underline{y_{kj}}^L + \underline{y_{kj}}^M + \underline{y_{kj}}^U}{6} \right) u_k \\
& \quad - \sum_{i=1}^{m} \left(\frac{\overline{x_{ij}}^L + 4\overline{x_{ij}}^M + \overline{x_{ij}}^U}{12} + \frac{\underline{x_{ij}}^L + \underline{x_{ij}}^M + \underline{x_{ij}}^U}{6} \right) v_i \leq 0, \ \forall j; j \neq 0 \\
\text{and } & u_k \geq 0, \quad k = 1, 2, \cdots, r, \\
& v_i \geq 0, \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{21}
$$

6. The super-efficiency score of the DMUs are calculated by solving the above crisp LP model of the bipolar fuzzy super efficiency model.

7. DMUs are ranked based on their super efficiency score. The DMU having highest super efficiency score rankled as 1 (or top) where as the DMUs with lowest super efficiency score is ranked as last.

8. The inefficient DMU can improve their efficiency score by identifying their corresponding peers or benchmarking units. These benchmarking units are the elements of the reference set. The reference set for each DMU can be calculated by following the given expression.

$$
\begin{aligned}
E_o =& \sum_{k=1}^{r} \left(\frac{\overline{y_{kj}}^L + 4\overline{y_{kj}}^M + \overline{y_{kj}}^U}{12} + \frac{\underline{y_{kj}}^L + \underline{y_{kj}}^M + \underline{y_{kj}}^U}{6} \right) u_k^* \\
=& \sum_{i=1}^{m} \left(\frac{\overline{x_{ij}}^L + 4\overline{x_{ij}}^M + \overline{x_{ij}}^U}{12} + \frac{\underline{x_{ij}}^L + \underline{x_{ij}}^M + \underline{x_{ij}}^U}{6} \right) v_i^*
\end{aligned} \tag{22}
$$

where, u_k^* and v_i^* are the optimal output and input weights for DMU$_o$, obtained by solving crisp LP model given in Eq. (20).

5 Numerical Example

This numerical example shows the assessment of ten DMUs in a bipolar fuzzy environment, with two inputs and three outputs represented as triangular bipolar fuzzy numbers to incorporate uncertainty in operational data, depicted in Table 1. This approach offers a thorough and realistic evaluation of efficiency by addressing the unpredictability and ambiguity associated with the performance assessment problem, allowing for comparison evaluations across DMUs with varying operational performance, ranking, and benchmarking results.

The outcomes derived from the BFDEA model, as illustrated in Table 2, furnish essential insights regarding the efficiency and comparative performance of the DMUs. Among the ten DMUs examined, four units ($D4, D5, D7$, and $D8$) have been identified as efficient, attaining an efficiency score of 1. This finding signifies that these DMUs are functioning on the efficiency frontier within the defined input-output framework in a bipolar fuzzy context. The remaining six DMUs ($D1, D2, D3, D6, D9$, and $D10$) are categorized as inefficient, exhibiting efficiency scores that fall short of 1. The efficiency score of the DMUs are compared in the Figure 3.

The super efficiency scores facilitate further distinction among the efficient DMUs, thereby allowing for a ranking predicated on their performance exceeding the efficiency frontier. DMU D5 is at the forefront with a super efficiency score of 1.4572, succeeded by $D7(1.4098)$ and $D8(1.3214)$. These scores accentuate the exceptional operational capabilities of D5, thereby establishing it as the preeminent unit. Such differentiation is imperative for identifying outstanding performers and establishing benchmarks for other units. Among the inefficient DMUs, $D1$, possessing an efficiency score of 0.9788, is relatively proximate to the efficiency frontier and occupies the fifth rank, whereas D9, with the lowest score of 0.3494, is positioned tenth. This disparity underscores the potential for enhancement in resource utilization and operational methodologies for the inefficient units. The rankings of inefficient DMUs yield significant insights into their relative performance and prospective areas for enhancement. DMUs such as $D6$ and $D9$, with efficiency scores of 0.3576 and 0.3494, respectively, manifest considerable inefficiencies, indicating a necessity for substantial operational modifications. Conversely, $D10$, with a score of 0.9196, is comparatively near to achieving efficiency and could attain this status with minimal improvements. These revelations suggest that focused strategies, including optimized resource allocation and enhanced input-output relationships, could significantly advance the performance of the inefficient DMUs.

Table 1. Triangular bipolar fuzzy inputs and outputs for efficiency measurement

DMU	Input 1	Input 2	Output 1	Output 2	Output 3
$D1$	$\langle 23, 27, 30; 21, 23, 27 \rangle$	$\langle 54, 61, 69; 47, 52, 58 \rangle$	$\langle 116, 125, 132; 112, 118, 124 \rangle$	$\langle 42, 47, 53; 40, 45, 50 \rangle$	$\langle 86, 92, 97; 84, 87, 91 \rangle$
$D2$	$\langle 17, 20, 24; 16, 22, 25 \rangle$	$\langle 71, 78, 83; 68, 74, 79 \rangle$	$\langle 93, 99, 106; 94, 96, 104 \rangle$	$\langle 34, 40, 45; 36, 45, 49 \rangle$	$\langle 51, 56, 64; 54, 58, 63 \rangle$
$D3$	$\langle 9, 12, 18; 8, 13, 16 \rangle$	$\langle 104, 108, 115; 105, 111, 118 \rangle$	$\langle 121, 128, 133; 124, 129, 131 \rangle$	$\langle 20, 26, 33; 18, 23, 27 \rangle$	$\langle 43, 48, 55; 41, 46, 51 \rangle$
$D4$	$\langle 11, 14, 18; 12, 16, 19 \rangle$	$\langle 64, 67, 72; 55, 58, 65 \rangle$	$\langle 128, 134, 141; 125, 135, 140 \rangle$	$\langle 55, 59, 65; 51, 54, 61 \rangle$	$\langle 74, 79, 88; 69, 75, 80 \rangle$
$D5$	$\langle 7, 11, 16; 9, 13, 17 \rangle$	$\langle 45, 57, 67; 49, 55, 60 \rangle$	$\langle 113, 118, 126; 115, 120, 125 \rangle$	$\langle 66, 73, 78; 57, 63, 69 \rangle$	$\langle 47, 53, 57; 43, 46, 50 \rangle$
$D6$	$\langle 20, 23, 25; 18, 21, 22 \rangle$	$\langle 120, 128, 135; 111, 117, 126 \rangle$	$\langle 76, 82, 86; 70, 75, 81 \rangle$	$\langle 25, 31, 37; 21, 26, 31 \rangle$	$\langle 35, 39, 46; 35, 42, 48 \rangle$
$D7$	$\langle 13, 15, 18; 14, 17, 21 \rangle$	$\langle 31, 38, 43; 36, 45, 50 \rangle$	$\langle 97, 102, 109; 101, 106, 115 \rangle$	$\langle 57, 64, 70; 52, 58, 66 \rangle$	$\langle 61, 68, 74; 58, 63, 69 \rangle$
$D8$	$\langle 7, 10, 13; 6, 8, 11 \rangle$	$\langle 84, 89, 95; 81, 87, 91 \rangle$	$\langle 118, 123, 129; 110, 115, 121 \rangle$	$\langle 33, 37, 44; 31, 36, 39 \rangle$	$\langle 49, 56, 61; 52, 58, 63 \rangle$
$D9$	$\langle 24, 26, 29; 21, 23, 27 \rangle$	$\langle 116, 120, 127; 108, 114, 119 \rangle$	$\langle 82, 87, 93; 80, 85, 92 \rangle$	$\langle 21, 26, 32; 24, 28, 32 \rangle$	$\langle 31, 36, 43; 25, 30, 36 \rangle$
$D10$	$\langle 10, 12, 16; 7, 11, 14 \rangle$	$\langle 87, 91, 98; 85, 90, 95 \rangle$	$\langle 120, 124, 130; 122, 128, 135 \rangle$	$\langle 37, 43, 49; 40, 45, 50 \rangle$	$\langle 42, 48, 55; 39, 45, 50 \rangle$

Table 2. Efficiency score and ranking of the DMUs

DMUs	Efficiency Score	Type	Super Efficiency Score	Ranking
$D1$	0.9788	Inefficient	0.9788	5
$D2$	0.5870	Inefficient	0.587	8
$D3$	0.8201	Inefficient	0.8201	7
$D4$	1.0000	Efficient	1.1332	4
$D5$	1.0000	Efficient	1.4572	1
$D6$	0.3576	Inefficient	0.3576	9
$D7$	1.0000	Efficient	1.4098	2
$D8$	1.0000	Efficient	1.3214	3
$D9$	0.3494	Inefficient	0.3494	10
$D10$	0.9196	Inefficient	0.9196	6

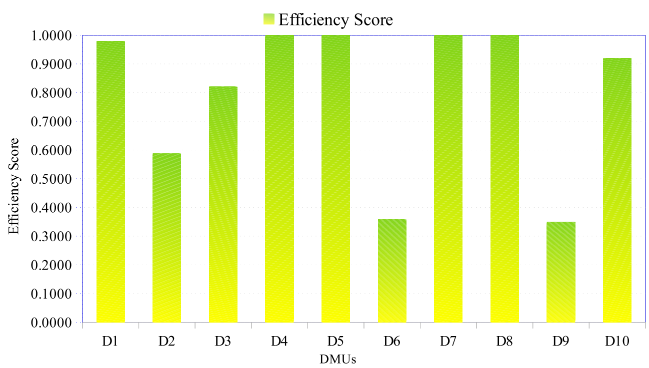

Figure 3. Efficiency score of the DMUs

The benchmarking evaluation of the DMUs given in Table 3, unveils critical perspectives on potential efficiency enhancements. The efficient units, namely $D4$, $D5$, $D7$, and $D8$, act as standards for their less efficient counterparts, signifying that these units function at optimal levels and may serve as valuable reference points. For example, $D1$ benchmarks itself against $D7$, implying that $D1$ ought to adopt methodologies or input-output configurations akin to those of $D7$ to boost its efficiency. $D2$, which is benchmarked against $D4$, $D5$, and $D7$, can achieve enhancements by harmonizing its operations with these proficient units. Furthermore, DMUs $D3$, $D6$, $D9$, and $D10$ also depend on the performance of efficient peers, underscoring the necessity for them to refine their processes in accordance with the highest-performing counterparts. The existence of multiple benchmarks for certain DMUs indicates varied trajectories for improvement, while self-benchmarking observed in $D4$, $D5$, $D7$, and $D8$ reaffirms their status as efficient entities. Hence, the analysis yields practical recommendations for underperforming DMUs to elevate productivity by emulating best practices established by their respective benchmark units.

Table 3. Benchmarking units for the DMUs

DMUs	Benchmarking Units
$D1$	$D7$
$D2$	$D4, D5, D7$
$D3$	$D5, D8$
$D4$	$D4$
$D5$	$D5$
$D6$	$D4, D5, D8$
$D7$	$D7$
$D8$	$D8$
$D9$	$D5, D8$
$D10$	$D8$

6 Conclusion

This study proposed a new bipolar fuzzy data envelopment analysis (BFDEA) framework that takes the advantage of BTFNs in the consideration of uncertainty and ambiguity in evaluating the performance of DMUs. The proposed framework, by establishing a value function for BTFNs and reformulating the BFDEA models in equivalent crisp linear programming forms, produces efficiency evaluations that are complete and accurate for both the including super-efficiency model and without it. The super-efficiency model provided additional detail about efficiently DMUs, ranking those from the efficiency frontier, thus allowing a complete ranking set on all DMUs. The developed benchmarking approaches provided identification for underperforming DMUs to their benchmark units which provided guidance for performance improvement. The numeric example demonstrated the example, and the relative robustness of the model to be generalizably applied to yield efficiency and benchmarking ratings that are more detailed and informative even with conflicting indicators commonly experienced in practice. The BFDEA method in this sense was clearly an improvement over traditional DEA models relative to its quantitative abilities to capture both positive and negative aspirations of uncertainty, again yielding superior insight and confidence for decision makers with its use in practice over the traditional DEA measures in the examples evaluated.

This study can be further expanded through future research using more complex types of bipolar fuzzy numbers such as the bipolar trapezoidal or general bipolar fuzzy sets to reflect the different forms of uncertainty. In addition, future research could develop more advanced solution algorithms that are guided by computational complexity and improve the scalability of the framework. Also, the application of the BFDEA framework to include stochastic conditions, or hybrid models that use fuzzy and probabilistic approach together, would further represent decision-making conditions. Finally, contributing to the BFDEA literature in various sectors, such as healthcare, energy, or supply chain management will support assessing the practical performance of the techniques and promote acceptance using BFDEA models for performance evaluation in real-world scenarios.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The author declares no conflict of interest.

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