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Numerical Solution of Time-Fractional Heat Conduction Problem in a Fuzzy Environment

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Abstract: The present paper emphasizes finding the solution for a fuzzy fractional heat conduction equation using the homotopy analysis transform method (HATM). The HATM combines two powerful, well-known methods: homotopy analysis method and the Laplace transform method. The approximate solution of the fuzzy fractional heat conduction equation is obtained by using HATM. Comparison with existing methods shows that the results obtained using the proposed method are in good agreement with the exact solutions available in the literature. All the numerical computations justify the proposed method is very efficient, effective, and simple for obtaining an approximate solution of the fuzzy time-fractional heat conduction equation.

Keywords: Laplace transform method; Caputo derivative; Homotopy analysis transform method (HATM); Time-fractional heat conduction equation with fuzziness; Approximate solution; Absolute error

1 Introduction

The fractional differential equations (FDEs) are the general version of the standard differential equations, where the order of the derivative is a non-integer order. They have many applications indifferent areas of science and technology, such as fluid mechanics, electrical networks, micro-electro mechanics systems, mass and heat transfer, visco- electricity, etc. These applications can be seen in studies [1–4].

In this paper, we have extended the idea to get the approximate analytical solutions of FDEs when uncertain behavior is observed in real phenomena. The idea of the fuzzy fractional differential equation was first introduced by Agarwal et al. [5], but later on this contribution was implemented by many authors to solve the fuzzy FDEs by using different analytical methods [6–8]. Some other important literature's for the fuzzy environment are found in studies [9–11].

We have used here a popular analytical method, namely the Homotopy analysis transform method (HATM), coupling of the Homotopy analysis method [12], and the Laplace transform method to solve the fuzzy fractional heat conduction equation. The detailed description of the HATM method has been discussed in studies [13–15]. The aim of this paper is to apply the HATM in a fuzzy environment to study the solution of the time-fractional heat conduction equation of the form:

$$\frac{\partial^\gamma \tilde{v}(x,t)}{\partial t^\gamma} = \delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} \tag{1}$$

where, $0 < \gamma \leq 1$ is the fractional derivative taken in Caputo sense [1–4].

The heat conduction Eq. (1) plays an important role in the modeling of enhancing the heat transfer between a solid surface and its convective, radioactive, or convective radioactive surface [16]. The main application of the Eq. (1) can be found in cylinders of aircraft engines, for cooling electric transformers, and other heat transfer equipment like nucleate boiling, convection, film boiling, and transition boiling. Recently, Eq. (1) has been studied using the homotopy perturbation method [16]. But as per the author's information, this is the first attempt for finding the approximate analytical solution of fuzzy fractional Eq. (1) by using the HATM.

The rest of the paper is organized as follows: In the next section, we have discussed the basic definitions and preliminaries related to fuzzy calculus and fractional derivative. The basic algorithm of the fractional homotopy analysis transform method in a fuzzy environment is discussed in Section 3. In Section 4, the application of the method to the fuzzy fractional heat conduction equation is discussed. Obtained results are discussed in Section 5. Then concludes the paper.

2 Basic Definitions of Fuzzy Calculus and Fractional Derivatives

Definition1. An ordered pair of functions n ordered pair of functions $(\underline{v}(\phi), \bar{v}(\phi)), 0 \leq \phi \leq 1)$ is said to be a fuzzy number $\tilde{v}$ if it satisfies the followings [6–8, 17–19]:

1. $\underline{v}(\phi)$ is bounded, left continuous and non-decreasing function on $[0, 1]$.
2. $\bar{v}(\phi)$ is bounded, left continuous and non-increasing function on $[0, 1]$.
3. $\underline{v}(\phi) \leq \bar{v}(\phi) \quad \forall \phi \in [0, 1]$.

Definition2. A fuzzy function is a mapping $v : T \to E^1, T \subseteq E^1$, with non-fuzzy variable and the ϕ set is defined as: $[v(t)]_\phi = [\underline{v}(t, \phi), \bar{v}(t, \phi)]$, where $t \in T$ and $0 \leq \phi \leq 1$ [6–8, 17–19].

Definition3. The Caputo fractional integral of order γ of fuzzy function $v(x, \phi)$ is defined as ${}_CD^{x_0}_\gamma v(x, \phi) = \left[{}_CD^{x_0}_\gamma \underline{v}(x, \phi), {}_CD^{x_0}_\gamma \bar{v}(x, \phi)\right]$, where [6–8, 17–19]

$${}_CD^{x_0}_\gamma \underline{v}(x) = \frac{1}{\Gamma(k-\gamma)} \int_x^{x_0} (x-t)^{k-\gamma-1} \underline{v}^k(t) dt = \frac{d^k v}{dt^k}, k-1 \leq \gamma < k, \gamma = k.$$

$${}_CD^{x_0}_\gamma \bar{v}(x) = \frac{1}{\Gamma(k-\gamma)} \int_x^{x_0} (x-t)^{k-\gamma-1} \bar{v}^k(t) dt = \frac{d^k \bar{v}}{dt^k}, k-1 \leq \gamma < k, \gamma = k.$$

Definition4. The below equation state about the Laplace transform of the fuzzy valued function $\tilde{v}(x, \phi)$ [6–8, 17–19]

$$\tilde{V}(x, \phi) = [\underline{V}(x, \phi), \bar{V}(x, \phi)]$$

$$\underline{V}(x, \phi) = L[\underline{v}(x)] = \int_0^\infty \exp^{-xt} \underline{v}(t) dt, t > 0$$

,

$$\bar{V}(x, \phi) = L[\bar{v}(x)] = \int_0^\infty \exp^{-xt} \bar{v}(t) dt, \quad t > 0.$$

3 Algorithm of Fractional Homotopy Analysis Transform Method (FHATM) in Fuzzy Environment

To discuss the basic algorithm of the FHATM, we take here time-fractional fuzzy heat conduction equation of the form:

$$\frac{\partial^\gamma \tilde{v}(x,t)}{\partial t^\gamma} = \delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} + \tilde{p}(x) \tag{2}$$

$x \in R, \quad 0 < x < 1, \quad t > 0, \quad \gamma \in (0, 1]$, where $\frac{\partial^\gamma \tilde{v}(x,t)}{\partial t^\gamma}$ is Caputo fractional derivative.

Operating Laplace transform on both side of the above equation, we get

$$L\left[\frac{\partial^\gamma \tilde{v}^(x,t)}{\partial t^\gamma}\right] - L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2}\right] = L[\tilde{p}(x)]. \tag{3}$$

By using Laplace transform of the Caputo fractional derivative,

$$r^\gamma L[\tilde{v}(x,t)] - \sum_{j=0}^{n-1} u^{\gamma-j-1} \tilde{v}^{(j)}(0,t) - L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2}\right] = L[\tilde{p}(x)] \tag{4}$$

or

$$L[\tilde{v}(x,t)] = r^{-\gamma} \sum_{j=0}^{n-1} r^{\gamma-j-1} \tilde{v}^{(j)}(0,t) + r^{-\gamma} L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} + \tilde{p}(x)\right]. \tag{5}$$

Next, the nonlinear operator is

$$\mathcal{T}[\tilde{\psi}(x,t;u)] = L[\tilde{\psi}(x,t;u)] - r^{-\gamma} \sum_{j=0}^{n-1} r^{\gamma-j-1} \tilde{v}^{(j)}(0,t) - r^{-\gamma} L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} + \tilde{p}(x)\right] \tag{6}$$

where, u is an auxiliary parameter from $[0,1]$ and $\tilde{\psi}(x,t;u)$ is real valued function of x,t and u.

The deformation equation of order zero of for the fuzzy fractional partial differential Eq. (2) is

$$(1-u)L\left[\tilde{\psi}(x,t;u)-\tilde{v}_0(x,t)\right]=huS(x,t)\mathcal{T}[\tilde{\psi}(x,t)] \tag{7}$$

where, $S(x,t)\neq 0$ an auxiliary function, $u\in[0,1]$, is an auxiliary parameter and h is a nonzero auxiliary parameter.

Here the unknown function $\tilde{\psi}(x,t;u)$ will be computed and initial approximate of $\tilde{v}(x,t)$ is $\tilde{v}_0(x,t)$. There has a freedom in FHATM to choose auxiliary parameter as well as initial approximation.

When $u=0, \tilde{\psi}(x,t;0)=\tilde{v}_0(x,t)$, when $u=1, \tilde{\psi}(x,t;1)=\tilde{v}(x,t)$.

As u varies from 0 to 1, solution $\tilde{\psi}(x,t;u)$ changes from initial guess $\tilde{v}_0(x,t)$ to the solution $\tilde{v}(x,t)$.

Using Taylor series expansion on $\tilde{\psi}(x,t;u)$ with respect to u we get,

$$\tilde{\psi}(x,t;u)=\tilde{v}_0(x,t)+\sum_{n=1}^{+\infty}\tilde{v}_n(x,t)u^n, \tag{8}$$

where

$$\tilde{v}_n(x,t)=\left[\frac{1}{\Gamma(n+1)}\frac{\partial^n\tilde{\psi}(x,t;u)}{\partial u^n}\right]_{u=0}. \tag{9}$$

The series converge at u =1, when auxiliary linear operator, initial guess, auxiliary parameter, and auxiliary function are chosen in properly.

$$\tilde{\psi}(x,t)=\tilde{v}_0(x,t)+\sum_{n=1}^{+\infty}\tilde{v}_n(x,t) \tag{10}$$

Now define vector as

$$\overrightarrow{\tilde{v}_l}=\{\tilde{v}_0(x,t),\tilde{v}_1(x,t),\ldots,\tilde{v}_l(x,t)\}. \tag{11}$$

Differentiating Eq. (7) n times with respect to parameter u and then by dividing $\Gamma(n+1)$ with the value of u =0 we get, Nth order deformation equation:

$$L\left[\tilde{v}_n(x,t)-\chi_n\tilde{v}_{n-1}(x,t)\right]=h\mathcal{T}(x,t)\rho_n\left(\overrightarrow{\tilde{v}}_{n-1},x,t\right), \tag{12}$$

where

$$\rho_n\left(\overrightarrow{\tilde{v}}_{n-1},x,t\right)=\frac{1}{\Gamma(n)}\left[\frac{\partial^{(n-1)}\mathcal{T}[\tilde{\psi}(x,t;u)]}{\partial u^{(n-1)}}\right]_{u=0}, \tag{13}$$

and $\chi_n=\begin{cases}0, & n\leq 1\\ 1, & n>1\end{cases}$

On Eq. (12), applying inverse Laplace transform, we get

$$\tilde{v}_n(x,t)=\chi_n\tilde{v}_{n-1}(x,t)]+L^{-1}\left[h\mathcal{T}(x,t)\rho_n\left(\overrightarrow{\tilde{v}}_{n-1},x,t\right)\right]. \tag{14}$$

Here,

$$\rho_n\left(\overrightarrow{\tilde{v}}_{n-1},x,t\right)=\frac{\partial^\gamma\tilde{v}_{n-1}(x,t)}{\partial t^\gamma}-\delta\frac{\partial^2\tilde{v}_{n-1}(x,t)}{\partial x^2}-(1-\chi_n)\tilde{p}(x). \tag{15}$$

In this manner, for $n\geq 1$ we get $\tilde{v}_n(x,t)$. For Nth order,

$$\tilde{v}(x,t)=\sum_{n=0}^{N}\tilde{v}_n(x,t) \tag{16}$$

We obtain accurate approximation of original fractional partial differential equation when $N \to \infty$. In case of FHATM in fuzzy environment there are two form of solutions.

The upper bound solution is

$$\bar{v}(x,t) = \sum_{n=0}^{\infty} \bar{v}_n(x,t) \tag{17}$$

and the lower bound solution is

$$\underline{v}(x,t) = \sum_{n=0}^{\infty} \underline{v}_n(x,t) \tag{18}$$

4 Application of the FHATM in Fuzzy Environment

Let us consider a fuzzy fractional heat conduction equation of the form [16]:

$$\frac{\partial^\gamma \tilde{v}(x,t)}{\partial t^\gamma} = \delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} \tag{19}$$

where, $x \in R, \quad 0 \leq x \leq 1, \quad t > 0$ and $\gamma \in (0,1]$.

Initial condition for the equation (4.1) is $\tilde{v}(x,0) = \tilde{w}(\phi)\sin 2\pi x, x \in [0,1]$ and the boundary condition is $\tilde{v}(0,t) = \tilde{v}(1,t) = 0$, where $\tilde{w}(\phi) = (\underline{w}(\phi), \overline{w}(\phi)) = (\phi - 1, 1 - \phi)\phi \in [0,1]$.

Here the exact solution is $\tilde{v}(x,t) = \tilde{w}(\phi)\sin(2\pi x) \cdot \exp\left(-4\pi^2 \delta t\right)$ [16].

By the similar manner as discussed in the above section employing FHATM method in fuzzy environment in the Eq. (19), we have

$$r^\gamma L[\tilde{v}(x,t)] - \sum_{j=0}^{n-1} r^{\gamma-j-1}\tilde{v}^{(j)}(0,t) - L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2}\right] = 0 \tag{20}$$

Next, the nonlinear operator for the Eq. (19) is:

$$\mathcal{T}[\tilde{\psi}(x,t;u)] = L[\tilde{\psi}(x,t;u)] - r^{-\gamma}\sum_{j=0}^{n-1} r^{\gamma-j-1}\tilde{v}^{(j)}(0,t) - r^{-\gamma} L\left[\delta \frac{\partial^2 \tilde{v}(x,t)}{\partial x^2}\right] \tag{21}$$

By taking $S(x,t) = 1$ and applying the process of FHATM, we get the values of the different approximations as follows:

$$\tilde{v}_0(x,t;\phi) = \tilde{w}(\phi)\sin 2\pi x$$
$$\tilde{v}_1(x,t;\phi) = -\frac{4h\delta\pi^2\tilde{w}(\phi)t^\gamma \sin 2\pi x}{\Gamma(\gamma+1)},$$
$$\tilde{v}_2(x,t;\phi) = -\frac{4h(h+1)\delta\pi^2\tilde{w}(\phi)t^\gamma \sin 2\pi x}{\Gamma(\gamma+1)} + \frac{16h^2\delta^2\pi^4\tilde{w}(\phi)t^{2\gamma}\sin 2\pi x}{\Gamma(2\gamma+1)}$$

and so on.

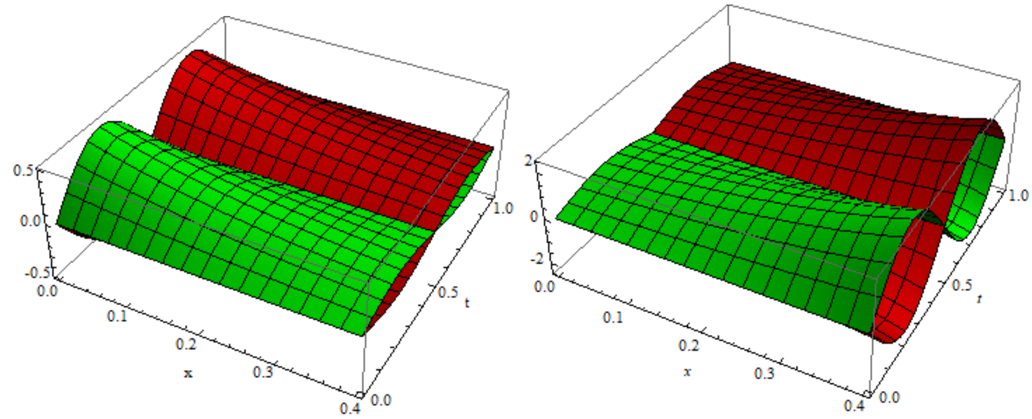
The approximate solution of the Eq. (19) is:

$$\tilde{v}(x,t) = \sum_{n=0}^{\infty} \tilde{v}_n(x,t) \tag{22}$$

and correspondingly we get the upper bound and lower bound solutions of the Eq. (19).

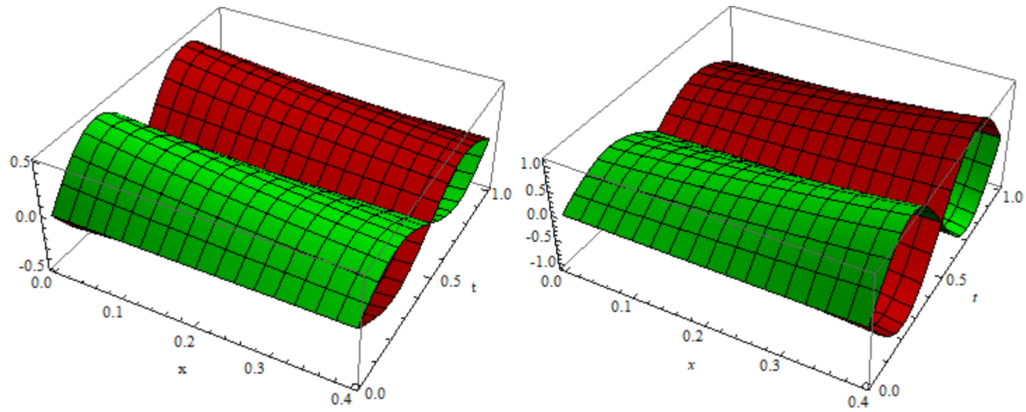
5 Results and Discussions

In this section, we discuss the efficiency and accuracy of obtained results by using HATM. Figure 1 and Figure 2 show the graphical comparison between the exact and approximate upper and lower bound solutions of Eq. (19) obtained by the fractional homotopy analysis transform method in a fuzzy environment. It is seen from Figure 1 and Figure 2, the upper and lower bound solutions obtained by the proposed method are clearly identical to the known exact solution when δ=0.1 and δ=0.05, respectively.



(a) (b)

Figure 1. Exact solution and approximate solution of upper and lower bound of the Eq. (19) graph when δ=0.1



(a) (b)

Figure 2. Exact solution and approximate solution of upper and lower bound of the Eq. (19) graph when δ=0.05

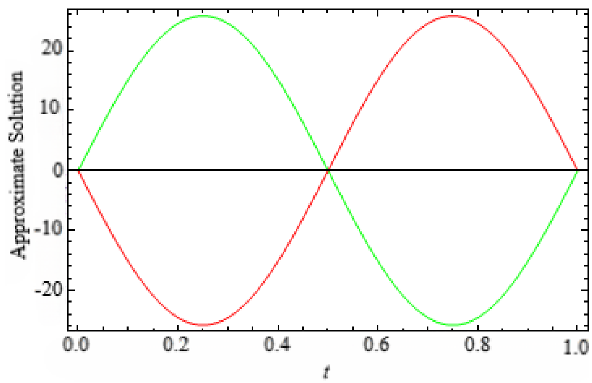


Figure 3. 2D plot of approximate solution of upper and lower bound of the Eq. (19)

Figure 3 represents the two-dimensional plot of upper and lower bound solutions of the Eq. (19). Figure 4 and Figure 5 are the two figures that represent the h-cut curve of upper and lower bound solutions, respectively. When γ takes the values 0.5, 0.75, 0.85, 0.95, and 1, we have curves pink, black, red, blue, and green, respectively. According to the h curve, there is freedom to choose the auxiliary parameter h. Here the acceptable range of h is $-2.20 \leq h \leq -0.7$. Figure 4 and Figure 5 show the line segments corresponding to the valid convergence region are nearly parallel to the horizontal axis. Next, Figure 6 and Figure 7 represent the absolute error curves of the upper bound and lower bound solutions of the Eq. (19). From Figure 6 and Figure 7 of absolute errors, it is seen that our obtained approximate solution by using the FHATM converges very rapidly to the exact solution. Finally, Table 1 and Table 2 give numerical values of upper bound solutions with the changes of values x and t when δ=0.05 and δ=0.1, respectively. Table 3 and Table 4 give numerical values of lower bound solutions with the changes of values x and t when δ=0.05 and δ=0.1, respectively.

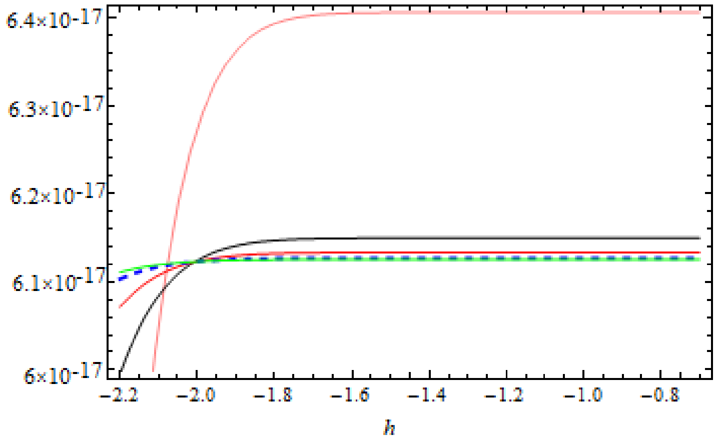


Figure 4. h-cut curve of approximate solution of upper bound of the Eq. (19)

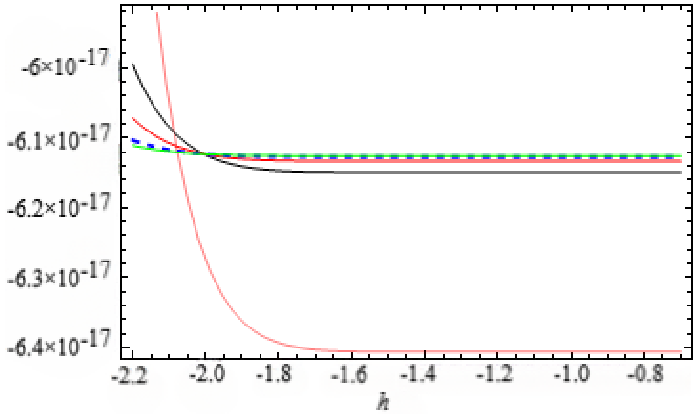


Figure 5. h-cut curve of approximate solution of lower bound of the Eq. (19)

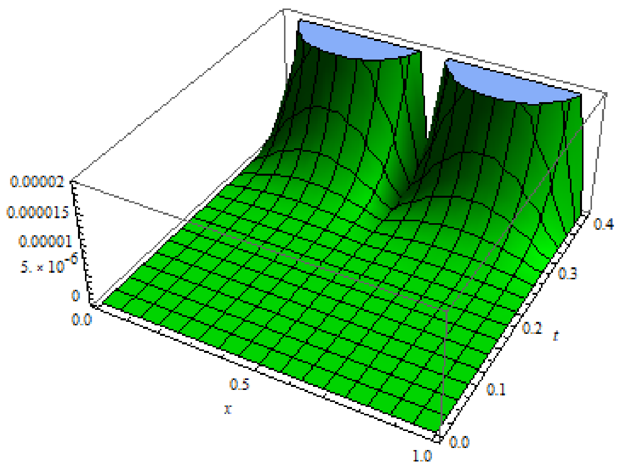


Figure 6. Absolute error curve of upper bound

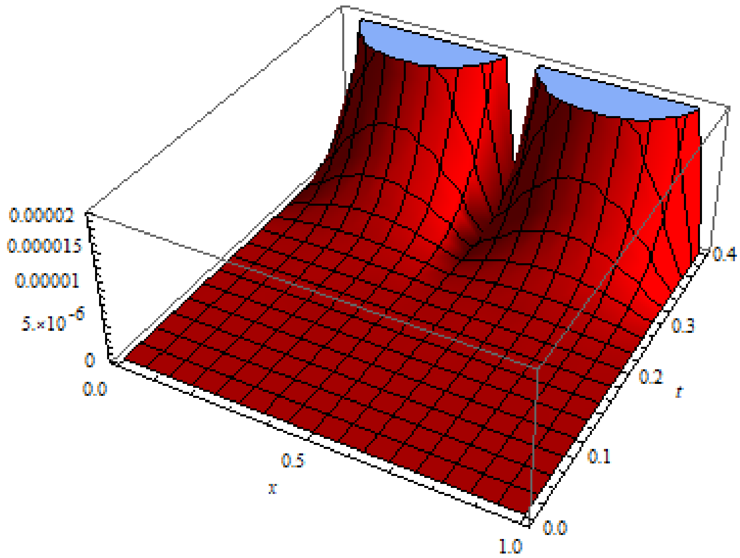


Figure 7. Absolute error curve of lower bound

Table 1. Numerical values of upper bound solution of Eq. (19) when δ=0.05

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0	0	0	0	0	0
0.05	0.154508	0.188226	0.2293	0.279339	0.340297
0.10	0.293893	0.358026	0.436155	0.531334	0.647283
0.15	0.404508	0.492781	0.600316	0.731318	0.890908
0.20	0.475528	0.579299	0.705714	0.859716	1.04733
0.25	0.5	0.609111	0.742032	0.903959	1.10122
0.30	0.475528	0.579299	0.705714	0.859716	1.04733
0.35	0.404508	0.492781	0.600316	0.731318	0.890908
0.40	0.293893	0.358026	0.436155	0.531334	0.647283
0.45	0.154508	0.188226	0.2293	0.279339	0.340297
0.50	$6.12323E^{-17}$	$7.45946E^{-17}$	$9.08727E^{-17}$	$1.10703E^{-16}$	$1.34861E^{-16}$
0.55	-0.154508	-0.188226	-0.2293	-0.279339	-0.340297
0.60	-0.293893	-0.358026	-0.436155	-0.531334	-0.647283
0.65	-0.404508	-0.492781	-0.600316	-0.731318	-0.890908
0.70	-0.475528	-0.579299	-0.705714	-0.859716	-1.04733
0.75	-0.5	-0.609111	-0.742032	-0.903959	-1.10122
0.80	-0.475528	-0.579299	-0.705714	-0.859716	-1.04733
0.85	-0.404508	-0.492781	-0.600316	-0.731318	-0.890908
0.90	-0.293893	-0.358026	-0.436155	-0.531334	-0.647283
0.95	-0.154508	-0.188226	-0.2293	-0.279339	-0.340297
1	0	0	0	0	0

Table 2. Numerical values of upper bound solution of Eq. (19) when δ=0.1

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0	0	0	0	0	0
0.05	0.154508	0.2293	0.340297	0.505022	0.749484
0.10	0.293893	0.436155	0.647283	0.960609	1.4256
0.15	0.404508	0.600316	0.890908	0.1 .32216	1.96217
0.20	0.475528	0.705714	1.04733	1.5543	2.30667
0.25	0.5	0.742032	1.10122	1.63428	2.42538
0.30	0.475528	0.705714	1.04733	1.5543	2.30667
0.35	0.404508	0.600316	0.890908	1.32216	1.96217

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0.40	0.293893	0.436155	0.647283	0.960609	1.4256
0.45	0.154508	0.2293	0.340297	0.505022	0.749484
0.50	6.12323E^{-17}	9.08727E^{-17}	1.34861E^{-16}	2.00142E^{-16}	2.97024E^{-16}
0.55	-0.154508	-0.2293	-0.340297	-0.505022	-0.749484
0.60	-0.293893	-0.436155	-0.647283	-0.960609	-1.4256
0.65	-0.404508	-0.600316	-0.890908	0.1 .32216	1.96217
0.70	0.475528	0.705714	1.04733	1.5543	2.30667
0.75	0.5	0.742032	1.10122	1.63428	2.42538
0.80	0.475528	0.705714	1.04733	1.5543	2.30667
0.85	0.404508	0.600316	0.890908	-1.32216	-1.96217
0.90	-0.293893	-0.436155	-0.647283	-0.960609	-1.4256
0.95	-0.154508	-0.2293	-0.340297	-0.505022	-0.749484
1	0	0	0	0	0

Table 3. Numerical values of lower bound solution of Eq. (19) when δ=0.05

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0	0	0	0	0	0
0.05	-0.154508	-0.188226	-0.2293	-0.279339	-0.340297
0.1	-0.293893	-0.358026	-0.436155	-0.531334	-0.647283
0.15	-0.404508	-0.492781	-0.600316	-0.731318	-0.890908
0.2	-0.475528	-0.579299	-0.705714	-0.859716	-1.04733
0.25	-0.5	-0.609111	-0.742032	-0.903959	-1.10122
0.30	-0.475528	-0.579299	-0.705714	-0.859716	-1.04733
0.35	-0.404508	-0.492781	-0.600316	-0.731318	-0.890908
0.40	-0.293893	-0.358026	-0.436155	-0.531334	-0.647283
0.45	-0.154508	-0.188226	-0.2293	-0.279339	-0.340297
0.50	-6.12323E^{-17}	-7.45946E^{-17}	-9.08727E^{-17}	-1.10703E^{-16}	-1.34861E^{-16}
0.55	0.154508	0.188226	0.2293	0.279339	0.340297
0.60	0.293893	0.358026	0.436155	0.531334	0.647283
0.65	0.404508	0.492781	0.600316	0.731318	0.890908
0.70	0.475528	0.579299	0.705714	0.859716	1.04733
0.75	0.5	0.609111	0.742032	0.903959	1.10122
0.80	0.475528	0.579299	0.705714	0.859716	1.04733
0.85	0.404508	0.492781	0.600316	0.731318	0.890908
0.90	0.293893	0.358026	0.436155	0.531334	0.647283
0.95	0.154508	0.188226	0.2293	0.279339	0.340297
1	0	0	0	0	0

Table 4. Numerical values of lower bound solution of Eq. (19) when δ=0.1

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0	0	0	0	0	0
0.05	-0.154508	-0.2293	-0.340297	-0.505022	-0.749484
0.10	-0.293893	-0.436155	-0.647283	-0.960609	-1.4256
0.15	-0.404508	-0.600316	-0.890908	0.1 .32216	1.96217
0.20	0.475528	0.705714	1.04733	1.5543	2.30667
0.25	0.5	0.742032	1.10122	1.63428	2.42538
0.30	0.475528	0.705714	1.04733	1.5543	2.30667
0.35	0.404508	0.600316	0.890908	-1.32216	-1.96217
0.40	-0.293893	-0.436155	-0.647283	-0.960609	-1.4256
0.45	-0.154508	-0.2293	-0.340297	-0.505022	-0.749484

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4
0.50	-6.12323E^{-17}	-9.08727E^{-17}	-1.34861E^{-16}	-2.00142E^{-16}	-2.97024E^{-16}
0.55	0.154508	0.2293	0.340297	0.505022	0.749484
0.60	0.293893	0.436155	0.647283	0.960609	1.4256
0.65	0.404508	0.600316	0.890908	0.1 .32216	1.96217
0.70	0.475528	0.705714	1.04733	1.5543	2.30667
0.75	0.5	0.742032	1.10122	1.63428	2.42538
0.80	0.475528	0.705714	1.04733	1.5543	2.30667
0.85	0.404508	0.600316	0.890908	1.32216	1.96217
0.90	0.293893	0.436155	0.647283	0.960609	1.4256
0.45	0.154508	0.2293	0.340297	0.505022	0.749484
1	0	0	0	0	0

6 Conclusions

We have used an effective iterative technique called the fractional homotopy analysis transform method in a fuzzy environment for finding the upper and lower solutions of the fuzzy time-fractional heat conduction equation. By using this method, we can both solve integer and fractional-order fuzzy differential equations. The fuzzy fractional differential equation is one of the emerging topics in the present era of research, where we found a beautiful combination of the fractional differential equation with fuzziness. To check the accuracy and exactness of the proposed method, we have illustrated different graphs. The value of error shows that there is a very small difference between exact and approximate solutions. Therefore, from the whole discussion, it may be concluded that the FHATM in a fuzzy environment is very powerful and efficient in finding approximate solutions as well as analytical solutions to many fuzzy fractional differential equations.

Author Contributions

Conceptualization, A.K. and S.M.; methodology, S.M.; software, S.M.; validation, A.K. and S.M.; formal analysis, A.K.; writing–original draft preparation, S.M.; writing–review and editing, A.K. and S.M.; supervision, A.K. All authors have read and agreed to the published version of the manuscript.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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